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Effect of BCS pairing on entrainment in neutron superfluid current in neutron star crust

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Abstract

The relative current density n^i of "conduction" neutrons in a neutron star crust beyond the neutron drip threshold can be expected to be related to the corresponding particle momentum covector p_i by a linear relation of the form $n^i = \mathcal{K}^{ij} p_j$ in terms of a physically well-defined mobility tensor $\mathcal{K}^{ij}$. This result is describable as an "entrainment" whose effect—wherever the crust lattice is isotropic—will simply be to change the ordinary neutron mass m to a "macroscopic" effective mass $m_\star$ such that in terms of the relevant number density n of unconfined neutrons we shall have $\mathcal{K}^{ij} = (n/m_\star)\gamma^{ij}$. In a preceding work based on a independent particle treatment beyond the Wigner–Seitz approximation, using Bloch type boundary conditions to obtain the distribution of energy $\mathcal{E}_k$ and associated group velocity $v_k^i = \partial\mathcal{E}_k/\partial\hbar k_i$ as a function of wave vector k_i, it was shown that the mobility tensor would be proportional to a phase space volume integral $\mathcal{K}^{ij} \propto \int \mathrm{d}^3k\, v_k^i v_k^j \delta\{\mathcal{E}_k - \mu\}$, where μ is the Fermi energy. Using the approach due to Bogoliubov, it is shown here that the effect of BCS pairing with a superfluid energy gap Δ_{F} and corresponding quasiparticle energy function $\in_k = \sqrt{(\mathcal{E}_k - \mu)^2 + \Delta_{\mathrm{F}}^2}$ will just be to replace the Dirac distributional integrand by the smoother distribution in the formula $\mathcal{K}^{ij} \propto \int \mathrm{d}^3k\, v_k^i v_k^j \Delta_{\mathrm{F}}^2/\in_k^3$. It is also shown how the pairing condensation gives rise to superfluidity in the technical sense of providing (meta) stability against resistive perturbations for a current that is not too strong (its momentum p_i must be small enough to give $2|p_i v_k^i| < \in_k^2/|\mathcal{E}_k - \mu|$ for all modes). It is concluded that the prediction of a very large effective

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mass enhancement in the middle layers of the star crust will not be significantly effected by the pairing mechanism.

1. Introduction

A two fluid model of neutron star cores, of the kind that is commonly used in hydrodynamical simulations has been developed in the past, assuming a neutron–proton–electron composition and using a nonrelativistic treatment, by the work of many authors [1,2]. In particular, as explained by Borumand et al. [3], appropriate expressions have been obtained for the relevant entrainment coefficients, relating the momentum of the neutron fluid to the particle current of both neutrons and protons, in terms of the Landau parameters in the Fermi liquid theory. More recently, it has been shown by Comer et al. [4,5] how the required entrainment coefficients, and the corresponding effective masses for the neutrons and the protons, can be evaluated in a relativistic treatment—such as is appropriate for deeper layers at densities substantially above the value (of about 10^{14} g/cm^3) that characterises ordinary nuclear matter—using a relativistic σ–ω mean field model. The salient conclusion to be drawn from all this work is that in these homogeneous fluid layers the effective mass of the neutron will be significantly but not enormously reduced (by several tens of percent) below its ordinary bare mass value.

The present article is part of a newer program of work [6,7] concerned with the previously unstudied problem of evaluating what turns out to be a much more substantial effective mass modification that is obtained from the corresponding entrainment coefficients in the crust at subnuclear densities, above the neutron drip threshold (at about 10^{11} g/cm^3). In these layers, relativistic corrections are insignificant but the issue is complicated by the microscopic inhomogeneity of the medium, in which some neutrons can still flow freely but protons are confined to atomic nuclei. As in the outer neutron star crust where all nucleons are bound, the nuclei of the inner crust will be liquid at high temperatures but will be held by Coulomb forces in a crystalline solid lattice in the relatively low temperature range (well below 10^8 degrees Kelvin) observed in ordinary isolated neutron stars, which according to theoretical considerations (see, e.g., [8]) should be attained within about a hundred years after the birth of the star. Such a low temperature regime will be maintained even in a binary system involving accretion provided its rate does not exceed the typical order of magnitude $\sim 10^{-10}$ $M_\odot$/yr [9]. It has long been generally recognised [10] that at such low temperatures (indeed all the way up to 10^9 degrees Kelvin or more [11]) the neutrons will form a BCS type condensate characterised by a superfluid energy gap, and—for the reasons discussed in the penultimate section of this article—will therefore be able to flow through the lattice for a macroscopically long time [12], without resistive or viscous dissipation. The evaluation of the entrainment coefficients for such a superfluid flow is of particular astrophysical interest, because relative motion of the effectively free "conduction" neutrons through the inner crust lattice is believed to be an essential element in the mechanism responsible for observed pulsar frequency "glitches".

In order to obtain the quantitative information that is needed, a first step has been the development by the present authors of a microscopic derivation [6] of an appropriate two fluid model for the inner crust regime, on the basis of a simplified nonrelativistic description of the underlying nuclear physics in which (as in the previously cited work [3,5] on the higher density regimes pertaining to deeper layers of the star) thermal corrections and the effects of superfluid pairing were neglected. Recent numerical work [7] has shown that this treatment implies an enormous enhancement by the entrainment of the effective mass of the free neutrons in the middle layers of the inner crust (a result that contrasts with the rather moderate diminution predicted for the effective neutron mass in the deeper core layers). The purpose of the present work is to evaluate the adjustments—which turn out to be small—that will result from allowance for the superfluid pairing of the neutrons.

Until now, quantum theoretical analysis of neutron superfluidity has mainly concentrated on static configurations (in either infinite medium [13] or inhomogeneous systems [14–16]), meaning states for which no current is actually flowing relative to crust. Even at densities substantially beyond the neutron drip threshold it should still be possible to obtain a reasonably accurate description for the static case by using the so-called Wigner–Seitz approximation that treats the neighbourhood of each ionic nucleus as if it were isolated in a sphere whose diameter is determined by the nearest neighbour distance. However for the treatment of more general—stationary but nonstatic—configurations involving relative conduction currents it is absolutely necessary to use a more realistic description in which the artificial Wigner–Seitz type boundary conditions are replaced by the natural Bloch type periodicity conditions that would be desirable for higher accuracy even in the static case.

The use of appropriate Bloch type periodicity conditions is routine in terrestrial solid state physics [17], but has so far been applied to neutron star matter only in a simplified zero temperature independent particle treatment [6,7] (of the kind applicable well below the Fermi temperature, but most appropriate for a young neutrons star that has not yet fallen below the critical temperature for the onset of superfluidity) for which the neutrons are considered to move as independent particles without allowance for the pairing interactions responsible for the superfluid energy gap that (in cool mature neutron stars) allows the currents to persist.

A simplified treatment of this kind has been used to show that the middle layers of a neutron star crust will be characterised by a very low value for the relevant mobility tensor in the formula $n^i = \mathcal{K}^{ij} p_j$ for the current $n^i = n\bar{v}^i$ of unbound neutrons (which will be present above the "drip" density of the order of 10^{11} g/cm^3) with number density n, mean velocity $\bar{v}^i$ and momentum per neutron p_i. Throughout this paper summation is understood over repeated covariant and contravariant coordinate Latin indices, for instance $\mathcal{K}^{ij} p_j \equiv \sum_j \mathcal{K}^{ij} p_j$. In the independent particle treatment, the mobility tensor was shown [6] to be given by a volume integral over the space of Bloch momentum covectors k_i that is expressible in terms of a Dirac distribution with support confined to the Fermi surface—where the relevant energy function $\mathcal{E}_{k\alpha}$ with a band index α, is equal to the chemical potential μ—in the form

$$\mathcal{K}^{ij} = \frac{2}{(2\pi)^3} \sum_\alpha \int v^i_{k\alpha} v^j_{k\alpha} \delta\{\mathcal{E}_{k\alpha} - \mu\}\, \mathrm{d}^3 k \tag{1}$$

(in which, as throughout this work, we use braces—as distinct from ordinary brackets—for functional dependence, in order to avoid possibly confusion with simple multiplication) where the relevant group velocity distribution is given by the usual formula

$$v^i_{k\alpha} = \frac{1}{\hbar}\frac{\partial \mathcal{E}_{k\alpha}}{\partial k_i}. \tag{2}$$

The main purpose of this article is to show how the preceding independent particle treatment can be generalised to allow for BCS type pairing using an approach of the kind pioneered by Bogoliubov. Since the relevant temperature range (substantially below 10^8 degrees Kelvin) is very small compared with the critical value (of the order of 10^9 degrees Kelvin) for the pairing condensation it will be justifiable for us to continue to use the zero temperature limit, $T = 0$, in which thermal correction effects are entirely ignored. One of the main motivations for this work is to check the robustness of the conclusions obtained from the simple treatment described above, particularly the prediction of a very low value for the mobility tensor, which is interpretable as meaning that the corresponding effective mass $m_\star = n/3\mathcal{K}^i{}_i$ will become very large compared with the ordinary neutron mass.

Our conclusion is that as a first step towards a more accurate treatment, in cases for which the superfluid pairing can be characterised just by a gap parameter $\Delta_{\rm F}$ the relevant integral over the Fermi surface will need to be replaced by a phase space volume integral given in terms of the quasiparticle energy

$$\euro_{k\alpha} = \sqrt{(\mathcal{E}_{k\alpha} - \mu)^2 + \Delta_{\rm F}^2} \tag{3}$$

by the new formula

$$\mathcal{K}^{ij} = \frac{2}{(2\pi)^3}\sum_\alpha \int v^i_{k\alpha} v^j_{k\alpha} \frac{\Delta_{\rm F}^2}{\euro_{k\alpha}}\,{\rm d}^3k, \tag{4}$$

in which the expression for the group velocity $v^i_{k\alpha}$ is the same as in the absence of the pairing gap. It is the diminution of this group velocity that is responsible for the enhancement of the effective mass, which on average should therefore not be greatly affected by the phase space smearing effect produced by the superfluid pairing. This comes from the fact that the expectation value in the superfluid phase of one particle quantities, namely the particle current density in the present work, are not very different from their normal (non superfluid) value as shown by Leggett [18] in the context of superfluid helium 3.

Although the effect of the pairing is not so important for the evaluation of the effective mass, it is of course important for the property of superfluidity itself. Much of the contemporary literature on the underlying mechanisms for “superconductivity” in astrophysical contexts deals only with purely static configurations, in which the essential question concerns the existence of a condensed state characterised by a finite energy gap. A secondary purpose of this article is to go back to the question of superconductivity in the strict technical sense, which refers not to static configurations but to stationary configurations involving the relevant motion of a current of some kind—electric in ordinary laboratory metals, but neutronic in the case of interest here. The essential issue is that of the (meta) stability of such a current against (small) perturbations of the kind that in the “normal” case would produce resistive damping. The original defining property of a superconductor is that it should be able to support a current that will be characterised by such metastability provided it does not exceed some finite critical value, beyond which "normal" dissipation will of course set in. It will be confirmed in the penultimate section of this article that within the framework of the simple theoretical model used for the present work this condition of superconductivity will indeed be satisfied, with a critical maximum current value that is estimated to be safely large compared with what is required for the relevant applications to astrophysical phenomena such as pulsar glitches.

2. Hamiltonian for the independent particle limit

The idea is to start on the basis of a second quantised formalism in terms of local fermionic field annihilation and creation operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ depending on space position coordinates x^i in a unit volume sample, and on a spin variable σ taking values $_\uparrow$ and $_\downarrow$ subject to anticommutation rules of the usual form

$$\big[\hat{\psi}_\sigma\{\mathbf{r}\}, \hat{\psi}_{\sigma'}\{\mathbf{r}'\}\big]_+ = 0, \qquad \big[\hat{\psi}^\dagger_\sigma\{\mathbf{r}\}, \hat{\psi}^\dagger_{\sigma'}\{\mathbf{r}'\}\big]_+ = 0, \tag{5}$$

$$\big[\hat{\psi}^\dagger_\sigma\{\mathbf{r}\}, \hat{\psi}_{\sigma'}\{\mathbf{r}'\}\big]_+ = \delta_{\sigma\sigma'}\delta\{\mathbf{r}, \mathbf{r}'\}, \tag{6}$$

using a quadratic Hamiltonian operator of the form

$$\hat{H} = \hat{H}_{\text{ind}} + \hat{H}_{\text{int}}, \quad \hat{H}_{\text{ind}} = \hat{H}_{\text{kin}} + \hat{H}_{\text{pot}}, \tag{7}$$

in which the interaction term $\hat{H}_{\text{int}}$ will be absent in the independent particle limit corresponding to the kind of model used [6] in our preceding first quantised treatment.

In this independent particle limit, only the kinetic and potential contributions are present and will be assumed to be given (neglecting possible spin dependence for simplicity) by integrals over the unit volume sample under consideration of the form

$$\hat{H}_{\text{kin}} = \int \mathrm{d}^3 r\, \hat{\mathcal{H}}_{\text{kin}}\{\mathbf{r}\}, \quad \hat{\mathcal{H}}_{\text{kin}}\{\mathbf{r}\} = \sum_\sigma \hat{\psi}^\dagger_\sigma\{\mathbf{r}\}\mathcal{H}_{\text{kin}}\hat{\psi}_\sigma\{\mathbf{r}\}, \tag{8}$$

$$\hat{H}_{\text{pot}} = \int \mathrm{d}^3 r\, \hat{\mathcal{H}}_{\text{pot}}\{\mathbf{r}\}, \quad \hat{\mathcal{H}}_{\text{pot}}\{\mathbf{r}\} = V\{\mathbf{r}\} \sum_\sigma \hat{\psi}^\dagger_\sigma\{\mathbf{r}\}\hat{\psi}_\sigma\{\mathbf{r}\}, \tag{9}$$

where $\mathcal{H}_{\text{kin}}$ is a self adjoint differential operator in the category specified in terms of a gauge covector a_i by an expression of the familiar form

$$\mathcal{H}_a = -\gamma^{ij}(\nabla_i + i a_i)\frac{1}{2m^\oplus\{\mathbf{r}\}}(\nabla_i + i a_i) \tag{10}$$

in which γ^{ij} is the Euclidean space metric and $m^\oplus\{\mathbf{r}\}$ is interpretable as a microscopic effective mass, which is usually found to have smaller values inside crustal nuclei [14]. The covector with components a_i is a gauge field allowing for the possibility of adjustment of the phases of the field operators $\hat{\psi}_\sigma\{\mathbf{r}\}$. In applications to particles with nonzero electric charge (e say) such as the electrons in an ordinary terrestrial superconductor or the protons in the deeper layers of a neutron star, the presence of such a field (taking the form $a_i = eA_i$) would be necessary for the treatment of magnetic effects, but in the uncharged case of the crust neutrons with which we are concerned here it will always be possible to work in the standard gauge for which this covector is simply set to zero, $a_i = 0$, which means that we simply take

$$\mathcal{H}_{\rm kin} = \mathcal{H}_0. \tag{11}$$

The potential $V\{\mathbf{r}\}$ and the microscopic effective mass $m^\oplus\{\mathbf{r}\}$ (as those deduced from contact two body interactions of the Skyrme type in the Hartree–Fock approximation) are supposed to represent the averaged effect on the neutrons of the nuclear medium and in particular of the ionic lattice. A periodic crystalline type lattice will be assumed, which implies that these functions should be invariant with respect to any lattice translation vectors

$$V\big\{\mathbf{r} + \ell^a \mathbf{e}_a\big\} = V\{\mathbf{r}\}, \qquad m^\oplus\big\{\mathbf{r} + \ell^a \mathbf{e}_a\big\} = m^\oplus\{\mathbf{r}\} \tag{12}$$

for any triad of integers ℓ^a ($a = 1, 2, 3$) in which the lattice basis vectors $\mathbf{e}_a$ may be interpreted as representing the interionic spacing in the solid case that will be relevant at very low temperature, but should in principle be taken to be much larger (so as to generate a giant cell interpretable as a typical mesoscopic average over a locally disordered configuration) for applications above the relevant melting temperature, at which it is to be expected that (unlike the weaker electron pairing mechanism in ordinary terrestrial superconductors) the superfluid neutron pairing mechanism will still be intact.

The mass function $m^\oplus\{\mathbf{r}\}$ in the specification of the kinetic contribution to the independent Hamiltonian will also be involved in the specification of the corresponding neutron current density operators, which will be given, for each value of the spin variable σ, by

$$\hat{n}^i_\sigma\{\mathbf{r}\} = \frac{\hbar}{2\mathrm{i} m^\oplus\{\mathbf{r}\}} \gamma^{ij} \big(\hat{\psi}^\dagger_\sigma\{\mathbf{r}\} \nabla_j \hat{\psi}_\sigma\{\mathbf{r}\} - \big(\nabla_j \hat{\psi}^\dagger_\sigma\big) \hat{\psi}_\sigma\{\mathbf{r}\}\big). \tag{13}$$

One of the main objectives of the present work is to obtain a practical way of evaluating the mean value of the total current, as given by the space averaged operator

$$\hat{\bar{n}}^i = \sum_\sigma \hat{\bar{n}}^i_\sigma, \quad \hat{\bar{n}}^i_\sigma = \int \mathrm{d}^3 r\, \hat{n}^i_\sigma, \tag{14}$$

as a function of the associated momentum in a stationary state that is nonstatic (and therefore nonisotropic, since the mean current will characterise a preferred direction) but uniform of a mesoscopic volume, meaning one that is large compared with the interionic spacing but small compared with the macroscopic lengthscales characterising the star crust thickness or even the intervortex separation.

It is to be noted for future reference that this current can be used to express the adjustment that will be required in cases when it turns out to be more convenient to work with the gauge adjusted operator $\mathcal{H}_a$ rather than $\mathcal{H}_0$ in the kinetic contribution (11): it can be seen that this kinetic contribution will be given in the small a limit by

$$\hat{H}_{\rm kin} + a_i \hat{\bar{n}}^i = \sum_\sigma \int \mathrm{d}^3 r\, \hat{\psi}^\dagger_\sigma\{\mathbf{r}\} \mathcal{H}_a \hat{\psi}_\sigma\{\mathbf{r}\} + \mathcal{O}\big\{|a|^2\big\}, \tag{15}$$

subject to the usual assumption that we are using periodic boundary conditions to get rid of a boundary term produced by an integration by parts using Green's theorem.

3. Representation by Bloch states

Subject to the usual Bloch type boundary conditions for a mesoscopic material sample of parallelepiped form—with a unit volume that is taken to be very large compared with the elementary lattice cells under consideration—the independent particle Hamiltonian will determine a complete orthonormal set of single particle states $\varphi_{k\alpha}\{\mathbf{r}\}$, labelled by a wave covector k_i taking discrete values on a fine mesh inside the first Brillouin zone and a band index α, satisfying the Floquet–Bloch theorem [17]

$$\varphi_{k\alpha}\{\mathbf{r}\} = u_{k\alpha}\{\mathbf{r}\}\mathrm{e}^{\mathrm{i}k\cdot\mathbf{r}}, \tag{16}$$

using the abbreviation $k\cdot\mathbf{r} = k_i x^i$, where $u_{k\alpha}\{\mathbf{r}\}$ satisfies the ordinary lattice periodicity conditions

$$u_{k\alpha}\left\{\mathbf{r} + \ell^a \mathbf{e}_a\right\} = u_{k\alpha}\{\mathbf{r}\}. \tag{17}$$

These wave functions are normalised as follows (using* to indicate complex conjugation):

$$\int \mathrm{d}^3 r\, \varphi^*_{k\alpha}\{\mathbf{r}\}\varphi_{l\beta}\{\mathbf{r}\} = \delta_{kl}\delta_{\alpha\beta}. \tag{18}$$

Setting the Bloch wave vector k_i in place of a_i in the definition (10) the eigenvalue equation can be usefully rewritten in terms of the ordinarily periodic functions $u_{k\alpha}$ as

$$(\mathcal{H}_k + V)u_{k\alpha}\{\mathbf{r}\} = \mathcal{E}_{k\alpha}u_{k\alpha}\{\mathbf{r}\}. \tag{19}$$

From the spin independence of the potential (by neglecting spin–orbit coupling terms), the phases can be chosen in such a way that we shall have

$$\varphi^*_{k\alpha}\{\mathbf{r}\} = \varphi_{-k\alpha}\{\mathbf{r}\}, \qquad u^*_{k\alpha}\{\mathbf{r}\} = u_{-k\alpha}\{\mathbf{r}\}. \tag{20}$$

These Bloch states may be employed in the usual way as a basis for the specification of corresponding position independent annihilation and creation operators, $\hat{c}_{\sigma\alpha k}$ and $\hat{c}^{\dagger}_{\sigma k\alpha}$, subject to anticommutation relations of the standard form

$$[\hat{c}_{\sigma k\alpha}, \hat{c}_{\sigma' l\beta}]_+ = 0, \qquad \left[\hat{c}^{\dagger}_{\sigma k\alpha}, \hat{c}^{\dagger}_{\sigma' l\beta}\right]_+ = 0, \tag{21}$$

$$\left[\hat{c}^{\dagger}_{\sigma k\alpha}, \hat{c}_{\sigma' l\beta}\right]_+ = \delta_{\sigma\sigma'}\delta_{kl}\delta_{\alpha\beta}, \tag{22}$$

in terms of which the original position dependent annihilation and creation operators will be given by

$$\hat{\psi}_\sigma\{\mathbf{r}\} = \sum_{k,\alpha}\varphi_{k\alpha}\{\mathbf{r}\}\hat{c}_{\sigma k\alpha}, \qquad \hat{\psi}^{\dagger}_\sigma\{\mathbf{r}\} = \sum_{k,\alpha}\varphi^*_{k\alpha}\{\mathbf{r}\}\hat{c}^{\dagger}_{\sigma k\alpha}. \tag{23}$$

It is evident just from the orthonormality conditions (18) that the spin dependent number density operator defined by

$$\hat{n}_\sigma\{\mathbf{r}\} = \hat{\psi}^{\dagger}_\sigma\{\mathbf{r}\}\hat{\psi}_\sigma\{\mathbf{r}\} \tag{24}$$

will have a space integral

$$\hat{\hat{n}}_\sigma = \int \mathrm{d}^3 r\, \hat{n}_\sigma\{\mathbf{r}\}, \tag{25}$$

(over the unit volume sample under consideration) that will be given by

$$\hat{\hat{n}}_\sigma = \sum_{k,\alpha} \hat{n}_{\sigma k\alpha}, \quad \hat{n}_{\sigma k\alpha} = \hat{c}^\dagger_{\sigma k\alpha} \hat{c}_{\sigma k\alpha}. \tag{26}$$

It can similarly be seen from the defining conditions (23) that integrated value of the independent particle contribution,

$$\hat{H}_{\text{ind}} = \int \mathrm{d}^3 r \, \hat{\mathcal{H}}_{\text{ind}}\{\mathbf{r}\}, \quad \hat{\mathcal{H}}_{\text{ind}}\{\mathbf{r}\} = \hat{\mathcal{H}}_{\text{kin}}\{\mathbf{r}\} + \hat{\mathcal{H}}_{\text{pot}}\{\mathbf{r}\}, \tag{27}$$

which is interpretable in the absence of the interaction contribution as the total energy operator, will be expressible in standard form as

$$\hat{H}_{\text{ind}} = \sum_{\sigma,k,\alpha} \mathcal{E}_{\alpha k} \hat{n}_{\sigma k\alpha}, \tag{28}$$

where $\hat{n}_{\sigma k\alpha}$ is the Bloch wave vector dependent particle number density operator given by (26).

To get an analogous formula for the mean current (over the unit volume sample under consideration) as given by the operator (14), we take its expectation value

$$\langle|\hat{\hat{n}}^i_\sigma|\rangle = \int \mathrm{d}^3 r \, \langle|\hat{n}^i_\sigma\{\mathbf{r}\}|\rangle, \tag{29}$$

for a state $|\rangle$ satisfying the simplicity condition that except for the diagonal contributions characterised by $\sigma' = \sigma$, $l_i = k_i$ and $\alpha = \beta$ the contributions of the expectation values $\langle|\hat{c}^\dagger_{\sigma k\alpha} \hat{c}_{\sigma' l\beta}|\rangle$ will vanish—or be negligible to the order of approximation under consideration—it can be seen that we shall obtain the formula

$$\langle|\hat{\hat{n}}^i_\sigma|\rangle = \sum_{k,\alpha} \langle|\hat{n}_{\sigma k\alpha}|\rangle v^i_{k\alpha}, \tag{30}$$

in which the relevant velocity will be given by

$$v^i_{k\alpha} = \int \mathrm{d}^3 r \, \frac{\hbar}{2\mathrm{i} m^\oplus\{\mathbf{r}\}} \gamma^{ij} \big(\varphi^*_{k\alpha}\{\mathbf{r}\} \nabla_j \varphi_{k\alpha}\{\mathbf{r}\} - \varphi_{k\alpha}\{\mathbf{r}\} \nabla_j \varphi^*_{k\alpha}\{\mathbf{r}\} \big). \tag{31}$$

This expression (31) for the velocity vector $v^i_{k\alpha}$ can easily be shown to be mathematically equivalent to the well known, albeit less intuitively obvious, group velocity formula that is given in terms of the single particle energy introduced in (19) by (2).

4. Characterisation of conducting reference state

The (zero temperature) states in which we are interested are those that minimise the expected total energy $\langle|\hat{H}|\rangle$ subject not only to the usual constraint that there should be a fixed given value of the corresponding total expected particle number

$$\langle|\hat{\hat{n}}|\rangle = \sum_\sigma \langle|\hat{\hat{n}}_\sigma|\rangle, \tag{32}$$

but also, since we are concerned with nonstatic—conducting—stationary configurations, to the requirement that there should also be a fixed given value of the expected total

$$\langle|\hat{\hat{n}}^i|\rangle = \sum_\sigma \langle|\hat{\hat{n}}^i_\sigma|\rangle, \tag{33}$$

of the current defined by (14).

Imposing these constraints by the introduction of corresponding Lagrange multipliers μ and p_i, the problem will effectively be that of unconstrained minimisation of the combination

$$\langle|\hat{H}'_{\{p\}}|\rangle = \langle|\hat{H}|\rangle - \mu\langle|\hat{\hat{n}}|\rangle - p_i\langle|\hat{\hat{n}}^i|\rangle, \tag{34}$$

in which we introduce the notation

$$\hat{H}'_{\{p\}} = \hat{H}' - p_i\hat{\hat{n}}^i, \quad \hat{H}' = \hat{H} - \mu\hat{\hat{n}}. \tag{35}$$

In the absence of the pair coupling term $\hat{H}_{\text{int}}$, the quantity to be minimised reduces to the form $\langle|\hat{H}'_{\text{ind}\{p\}}|\rangle$ with

$$\hat{H}'_{\text{ind}\{p\}} = \hat{H}'_{\text{ind}} - p_i\hat{\hat{n}}^i, \quad \hat{H}'_{\text{ind}} = \hat{H}_{\text{ind}} - \mu\hat{\hat{n}}. \tag{36}$$

It can be seen from (11) and (15) that, for a small value,

$$p_i = \hbar q_i, \tag{37}$$

of the momentum, its effect will be given to first order in the magnitude $|q|$ of the corresponding wave number covector just by substituting the gauge adjusted operator $\mathcal{H}_{-q}$ in place of $\mathcal{H}_0$ in the relevant differential formulae. Thus, in particular, it can be seen that the appropriate modification of the eigenvalue equation (19) for the required replacements $\mathcal{E}_{\{p\}k\alpha}$ and $u_{\{p\}k\alpha}$ of $\mathcal{E}_{k\alpha}$ and $u_{k\alpha}$ will be just the substitution of $\mathcal{H}_{k-q}$ for $\mathcal{H}_k$, which evidently means that to this first order of accuracy we shall have

$$\mathcal{E}_{\{p\}k\alpha} = \mathcal{E}_{(k-q)\alpha}, \tag{38}$$

and

$$u_{\{p\}k\alpha} = u_{(k-q)\alpha}. \tag{39}$$

One thereby obtains the formula

$$\varphi_{\{p\}k\alpha}\{\mathbf{r}\} = \mathrm{e}^{\mathrm{i}q\cdot\mathbf{r}}\varphi_{(k-q)\alpha}\{\mathbf{r}\}, \tag{40}$$

for the corresponding modification of the single particle states (16), which in turn, by the analogue of (23), determine correspondingly adjusted annihilation and creation operators, $\hat{c}_{\{p\}\sigma k\alpha}$ and $\hat{c}^\dagger_{\{p\}\sigma k\alpha}$, in terms of which (26) can be rewritten in the equivalent form

$$\hat{\hat{n}}_\sigma = \sum_{k,\alpha}\hat{n}_{\{p\}\sigma k\alpha}, \quad \hat{n}_{\{p\}\sigma k\alpha} = \hat{c}^\dagger_{\{p\}\sigma k\alpha}\hat{c}_{\{p\}\sigma k\alpha}. \tag{41}$$

It can thus be seen that, as the analogue of (28), the effective Hamiltonian $\hat{H}'_{\{p\}\text{ind}}$ will be given by the formula

$$\hat{H}'_{\{p\}\text{ind}} = \sum_{\sigma,k,\alpha}\mathcal{E}'_{\{p\}k\alpha}\hat{n}_{\{p\}k\alpha}, \tag{42}$$

with

$$\mathcal{E}'_{\{p\}k\alpha} = \mathcal{E}'_{(k-q)\alpha}, \tag{43}$$

to the linear order accuracy in p with which we are working. At this order, it can be seen from (2) that (43) may be rewritten as

$$\mathcal{E}'_{\{p\}k\alpha} = \mathcal{E}'_{k\alpha} - p_i v^i_{k\alpha}, \quad \mathcal{E}'_{k\alpha} = \mathcal{E}_{k\alpha} - \mu. \tag{44}$$

The expectation value of the quantity given by (42) will evidently be minimised by a reference state vector $|\rangle = |_{\{\mu,p\}}\rangle$ that is chosen (as a function of the multipliers μ and p_i) in such a way that the expectation $\langle_{\{\mu,p\}}|\hat{n}_{\{p\}\sigma k\alpha}|_{\{\mu,p\}}\rangle$ has its maximum value, namely 1, whenever $\mathcal{E}'_{\{p\}k\alpha}$ is negative, and its minimum value, namely zero, whenever $\mathcal{E}'_{\{p\}k\alpha}$ is positive. It can thus be seen from (43) that the effect of the current will consist just of a uniform shift of the distribution in momentum space by an amount given by the infinitesimal momentum covector p_i. Such a state is characterised by the conditions

$$\hat{n}_{\{p\}\sigma k\alpha}|_{\{\mu,p\}}\rangle = n_{\{p\}\sigma k\alpha}|_{\{\mu,p\}}\rangle \tag{45}$$

with the eigenvalues given as a Heaviside distribution by

$$n_{\{p\}\sigma k\alpha} = \vartheta\{-\mathcal{E}'_{\{p\}k\alpha}\}. \tag{46}$$

It can be seen that this state $|_{\{\mu,p\}}\rangle$ satisfies the condition for applicability of the analogue of (30), and hence that the expected mean current value

$$\bar{n}^i_\sigma = \langle_{\{\mu,p\}}|\hat{\bar{n}}^i_\sigma|_{\{\mu,p\}}\rangle \tag{47}$$

will be given (in accordance with our previous evaluation [6] in a first quantised framework) by

$$\bar{n}^i_\sigma = \sum_{k,\alpha} v^i_{k\alpha}\,\vartheta\{-\mathcal{E}'_{\{p\}k\alpha}\}. \tag{48}$$

In the linearised weak current limit with which we are working, it can be seen from (2) and (44) that this will be expressible to first order in terms of the static limit value,

$$n_{\sigma k\alpha} = \vartheta\{\mu - \mathcal{E}_{k\alpha}\}, \tag{49}$$

of the distribution (46) as

$$\bar{n}^i_\sigma = p_j \sum_{k,\alpha} \frac{n_{\sigma k\alpha}}{\hbar^2}\frac{\partial^2 \mathcal{E}_{k\alpha}}{\partial k_i \partial k_j}. \tag{50}$$

5. Bogoliubov treatment of pairing

Up to this point what has been done in the present article is just to translate the work of our preceding article [6] from first quantised to second quantised formalism. The motivation for this translation is that a second quantised treatment is indispensable for the next step, which is to go beyond the independent particle model used in the preceding work by including allowance for pairing interactions.

In the inner crust in which we are concerned here, dripped neutrons are expected to be paired in spin singlet states [10], as the usual Cooper pairs of electrons in terrestrial superconductors. A standard way of allowing for pairing interaction in a mean field model (assuming for simplicity a contact two body interaction as it is the case for conventional superconductor [19] and a common practice in nuclear physics [20]) is thus to take the interaction contribution in (7) to have the form

$$\hat{\mathcal{H}}_{\rm int}\{\mathbf{r}\} = \Delta\{\mathbf{r}\}\hat{\psi}^{\dagger}_{\uparrow}\{\mathbf{r}\}\hat{\psi}^{\dagger}_{\downarrow}\{\mathbf{r}\} + \Delta^{*}\{\mathbf{r}\}\hat{\psi}_{\downarrow}\{\mathbf{r}\}\hat{\psi}_{\uparrow}\{\mathbf{r}\}, \tag{51}$$

where $\Delta\{\mathbf{r}\}$ is a position dependent complex potential that, in a "self consistent" model, should be expressible in terms of the abnormal density expectation value $\langle|\hat{\psi}_{\downarrow}\{\mathbf{r}\}\hat{\psi}_{\uparrow}\{\mathbf{r}\}|\rangle$ in the relevant reference state $|\rangle$.

The mean complex phase of the function $\Delta\{\mathbf{r}\}$ is subject to an indeterminacy that can be resolved by fixing the phase in the specification of the wave operators. In a static configuration one would expect that this coupling potential $\Delta\{\mathbf{r}\}$ would share the ordinary lattice periodicity property (12) and moreover that the phase should be adjustable in such a way as to ensure that Δ becomes real.

Instead of using the representation (16) in terms of the simple Bloch wave functions $\varphi_{k\alpha}\{\mathbf{r}\}$, in the approach introduced by Bogoliubov one seeks a more general representation whereby the single component Bloch waves are replaced by two component Bloch functions with components $\varphi^{0}_{k\alpha}\{\mathbf{r}\}$ and $\varphi^{1}_{k\alpha}\{\mathbf{r}\}$ that are characterised with respect to corresponding ordinarily periodic functions $u^{0}_{k\alpha}\{\mathbf{r}\}$, and $u^{1}_{k\alpha}\{\mathbf{r}\}$ by

$$\begin{pmatrix} \varphi^{0}_{k\alpha}\{\mathbf{r}\} \\ \varphi^{1}_{k\alpha}\{\mathbf{r}\} \end{pmatrix} = \mathrm{e}^{\mathrm{i}k\cdot\mathbf{r}} \begin{pmatrix} u^{0}_{k\alpha}\{\mathbf{r}\} \\ u^{1}_{k\alpha}\{\mathbf{r}\} \end{pmatrix}. \tag{52}$$

These functions are used for replacing the original representation (16) by a mixed particle–hole representation involving a new set of position independent quasiparticle annihilation and creation operators $\hat{\gamma}_{\sigma k}$ and $\hat{\gamma}^{\dagger}_{\sigma k}$ in terms of which we shall have

$$\hat{\psi}_{\uparrow}\{\mathbf{r}\} = \sum_{k,\alpha}\bigl(\varphi^{0}_{k\alpha}\{\mathbf{r}\}\hat{\gamma}_{\uparrow k\alpha} - \varphi^{1*}_{k\alpha}\{\mathbf{r}\}\hat{\gamma}^{\dagger}_{\downarrow k\alpha}\bigr), \tag{53}$$

and

$$\hat{\psi}_{\downarrow}\{\mathbf{r}\} = \sum_{k}\bigl(\varphi^{0}_{k\alpha}\{\mathbf{r}\}\hat{\gamma}_{\downarrow k\alpha} + \varphi^{1*}_{k\alpha}\{\mathbf{r}\}\hat{\gamma}^{\dagger}_{\uparrow k\alpha}\bigr), \tag{54}$$

where the new operators satisfy anticommutation relations of the standard form

$$[\hat{\gamma}_{\sigma k\alpha}, \hat{\gamma}_{\sigma' l\beta}]_{+} = 0, \qquad \bigl[\hat{\gamma}^{\dagger}_{\sigma k\alpha}, \hat{\gamma}^{\dagger}_{\sigma' l\beta}\bigr]_{+} = 0, \tag{55}$$

$$\bigl[\hat{\gamma}^{\dagger}_{\sigma k\alpha}, \hat{\gamma}_{\sigma' l\beta}\bigr]_{+} = \delta_{\sigma\sigma'}\delta_{kl}\delta_{\alpha\beta}. \tag{56}$$

As a result, consistency with (5) and (6) entails the relations:

$$\bigl[\hat{\psi}^{\dagger}_{\uparrow}\{\mathbf{r}\}, \hat{\psi}_{\uparrow}\{\mathbf{r}'\}\bigr]_{+} = \delta\{\mathbf{r},\mathbf{r}'\} = \sum_{k}\varphi^{0*}_{k\alpha}\{\mathbf{r}\}\varphi^{0}_{k\beta}\{\mathbf{r}'\} + \varphi^{1*}_{k\alpha}\{\mathbf{r}'\}\varphi^{1}_{k\alpha}\{\mathbf{r}\}, \tag{57}$$

$$\bigl[\hat{\psi}^{\dagger}_{\downarrow}\{\mathbf{r}\}, \hat{\psi}^{\dagger}_{\uparrow}\{\mathbf{r}'\}\bigr]_{+} = 0 = \sum_{k}\varphi^{1}_{k\alpha}\{\mathbf{r}\}\varphi^{0*}_{k\alpha}\{\mathbf{r}'\} - \varphi^{0*}_{k\alpha}\{\mathbf{r}\}\varphi^{1}_{k\alpha}\{\mathbf{r}'\}. \tag{58}$$

The purpose of such a Bogoliubov ansatz is to enable us to choose the new functions $\varphi^0_{k\alpha}\{\mathbf{r}\}$ and $\varphi^1_{k\alpha}\{\mathbf{r}\}$ in such a way as to simplify the expression for the total effective Hamiltonian, which will be given for a static configuration by

$$\hat{H}' = \int \mathrm{d}^3 r\, \hat{\mathcal{H}}'\{\mathbf{r}\}, \tag{59}$$

with

$$\hat{\mathcal{H}}'\{\mathbf{r}\} = \sum_\sigma \hat{\psi}^\dagger_\sigma\{\mathbf{r}\}\mathcal{H}'_{\mathrm{ind}}\hat{\psi}_\sigma\{\mathbf{r}\} + \Delta\{\mathbf{r}\}\hat{\psi}^\dagger_\uparrow\{\mathbf{r}\}\hat{\psi}^\dagger_\downarrow\{\mathbf{r}\} + \Delta^*\{\mathbf{r}\}\hat{\psi}_\downarrow\{\mathbf{r}\}\hat{\psi}_\uparrow\{\mathbf{r}\}, \tag{60}$$

in which the independent particle contribution is given by

$$\mathcal{H}'_{\mathrm{ind}} = \mathcal{H}_{\mathrm{ind}} - \mu, \tag{61}$$

where, as before, μ is a Lagrange multiplier, whose purpose when we apply the variation principle, is to impose the constraint that the expectation of the total integrated number density should be held fixed. It is to be remarked that in the presence of the pairing interaction term, the number operator $\hat{\hat{n}}$ will no longer exactly commute with the Hamiltonian, which implies that the state that minimises the expectation of the effective Hamiltonian obtained in this way will not be an exact eigenstate either of the particle number or of the energy.

The simplification of the Hamiltonian (60) can be achieved by taking the functions $\varphi^0_{k\alpha}\{\mathbf{r}\}$ and $\varphi^1_{k\alpha}\{\mathbf{r}\}$ to be solutions of the coupled set of differential equations (known as the Bogoliubov–de Gennes equations in the condensed matter field [21]) given by

$$\begin{pmatrix} \mathcal{H}'_{\mathrm{ind}} & \Delta \\ \Delta^* & -\mathcal{H}'^*_{\mathrm{ind}} \end{pmatrix} \begin{pmatrix} \varphi^0_{k\alpha} \\ \varphi^1_{k\alpha} \end{pmatrix} = \epsilon_{k\alpha} \begin{pmatrix} \varphi^0_{k\alpha} \\ \varphi^1_{k\alpha} \end{pmatrix}, \tag{62}$$

in which the eigenvalue $\epsilon_{k\alpha}$ is what will be seen to be interpretable as the relevant quasiparticle energy. This system can be written more explicitly in terms of the ordinarily periodic functions $u^0_{k\alpha}\{\mathbf{r}\}$, and $u^1_{k\alpha}\{\mathbf{r}\}$ introduced in (52) as

$$\begin{pmatrix} \mathcal{H}_k + V' & \Delta \\ \Delta^* & -\mathcal{H}^*_k - V' \end{pmatrix} \begin{pmatrix} u^0_{k\alpha} \\ u^1_{k\alpha} \end{pmatrix} = \epsilon_{k\alpha} \begin{pmatrix} u^0_{k\alpha} \\ u^1_{k\alpha} \end{pmatrix}, \tag{63}$$

using the notation of (61), where

$$V' = V - \mu. \tag{64}$$

The foregoing specification is incomplete, because the condition of satisfying (62) will evidently be preserved by interchanges of the form

$$\varphi^{1*}_{k\alpha} \leftrightarrow \varphi^0_{-k\alpha}, \qquad \epsilon_{k\alpha} \leftrightarrow -\epsilon_{k\alpha}, \tag{65}$$

but this ambiguity is resolved by adoption of the usual postulate that the eigenvalues be positive,

$$\epsilon_{k\alpha} > 0. \tag{66}$$

To fix the normalisation of the solutions, which will automatically satisfy the integral relations expressible—restoring the explicit reference to the position dependence—as

$$\int \mathrm{d}^3 r\, \varphi^1_{k\alpha}\{\mathbf{r}\}\varphi^0_{l\beta}\{\mathbf{r}\} = \int \mathrm{d}^3 r\, \varphi^0_{k\alpha}\{\mathbf{r}\}\varphi^1_{l\beta}\{\mathbf{r}\}, \tag{67}$$

the amplitude of the (automatically mutually orthogonal) solutions is chosen so that

$$\int \mathrm{d}^3 r \left(\varphi^{0*}_{k\alpha}\{\mathbf{r}\}\varphi^{0}_{l\beta}\{\mathbf{r}\} + \varphi^{1*}_{k\alpha}\{\mathbf{r}\}\varphi^{1}_{l\beta}\{\mathbf{r}\}\right) = \delta_{kl}\delta_{\alpha\beta}. \tag{68}$$

The foregoing ansatz eliminates all the undesirable terms, reducing the effective Hamiltonian operator to the remarkably simple form

$$\hat{H}' = \sum_{\sigma,k,\alpha} \in_{k\alpha}\left(\hat{\gamma}^{\dagger}_{\sigma k\alpha}\hat{\gamma}_{\sigma k\alpha} - \sin^2\theta_{k\alpha}\right) = \sum_{\sigma,k\,\alpha} \in_{k\alpha}\left(\cos^2\theta_{k\alpha} - \hat{\gamma}_{\sigma k\alpha}\hat{\gamma}^{\dagger}_{\sigma k\alpha}\right), \tag{69}$$

in which $\theta_{k\alpha}$ is the relevant Bogoliubov angle, as defined, for each value of the wavenumber covector k_i and band index α by

$$\cos^2\theta_{k\alpha} = \int \mathrm{d}^3 r\, \varphi^{0*}_{k\alpha}\{\mathbf{r}\}\varphi^{0}_{k\alpha}\{\mathbf{r}\}, \qquad \sin^2\theta_{k\alpha} = \int \mathrm{d}^3 r\, \varphi^{1*}_{k\alpha}\{\mathbf{r}\}\varphi^{1}_{k\alpha}\{\mathbf{r}\}. \tag{70}$$

By minimisation of the expectation of the operator (69) one obtains the required condensate reference state $|\rangle = |_{\{\mu\}}\rangle$, which is characterised by the condition

$$\hat{\gamma}_{\sigma k\alpha}|_{\{\mu\}}\rangle = 0, \tag{71}$$

expressing absence of all the quasiparticles created by the operators $\hat{\gamma}^{\dagger}_{\sigma k\alpha}$.

The quasiparticle operators can be written in terms of the particle operators remembering equation (23) and the orthonormality condition (18) as

$$\hat{c}_{\uparrow k\alpha} = \sum_{l,\beta}\left(U_{k\alpha,l\beta}\hat{\gamma}_{\uparrow l\beta} - V_{k\alpha,l\beta}\hat{\gamma}_{\downarrow}{}^{\dagger}_{l\beta}\right), \tag{72}$$

$$\hat{c}_{\downarrow k\alpha} = \sum_{l\beta}\left(U_{k\alpha,l\beta}\hat{\gamma}_{\downarrow l\beta} + V_{k\alpha,l\beta}\hat{\gamma}_{\uparrow}{}^{\dagger}_{l\beta}\right), \tag{73}$$

where we have introduced the matrices

$$U_{k\alpha,l\beta} = \int \mathrm{d}^3 r\, \varphi^{*}{}_{k\alpha}\{\mathbf{r}\}\varphi^{0}_{l\beta}\{\mathbf{r}\}, \tag{74}$$

$$V_{k\alpha,l\beta} = \int \mathrm{d}^3 r\, \varphi^{*}{}_{k\alpha}\{\mathbf{r}\}\varphi^{1*}_{l\beta}\{\mathbf{r}\}, \tag{75}$$

which from the properties of Bloch wave functions reduce to

$$U_{k\alpha,l\beta} = \delta_{kl}U_{k\alpha,k\beta}, \qquad V_{k\alpha,l\beta} = \delta_{-kl}V_{k\alpha,-k\beta}. \tag{76}$$

It is to be noted that $\varphi_{k\alpha}$, $\varphi^{0}_{k\alpha}$ and $\varphi^{1}_{k\alpha}$ are all Bloch wave functions associated with the same Bloch wave vector (hence having the same phase shift whenever translated from one cell to another) but are solutions of different equations. Inserting these expressions into the number density operator $\hat{n}_{\sigma k\alpha}$ introduced in (26), it is readily verified that its expectation value in the superfluid ground state is given by

$$\langle_{\{\mu\}}|\hat{n}_{\sigma k\alpha}|_{\{\mu\}}\rangle = \sum_{\beta}|V_{k\alpha,-k\beta}|^2. \tag{77}$$

Remembering that $\varphi_{k\alpha}$ are the single particle states of the independent Hamiltonian (61), with energies $\mathcal{E}'_{k\alpha}$, it can be seen from the Bogoliubov equations (62) that the expression (77) is equivalent to

$$\langle_{\{\mu\}}|\hat{n}_{\sigma k\alpha}|_{\{\mu\}}\rangle = \sum_{\beta} \frac{|\Delta^0_{k\alpha,-k\beta}|^2}{(\epsilon_{-k\beta} + \mathcal{E}'_{k\alpha})^2}, \tag{78}$$

where

$$\Delta^0_{k\alpha,l\beta} = \int \mathrm{d}^3r\, \varphi^*_{k\alpha}\{\mathbf{r}\}\Delta\{\mathbf{r}\}\varphi^{0*}_{l\beta}\{\mathbf{r}\}. \tag{79}$$

6. The BCS ansatz

Since (particularly for the middle layers of a neutron star crust, where the effective mass enhancement is likely [6,7] to be most important) we are still far from having a sufficient knowledge of the solutions $\varphi_{k\alpha}\{\mathbf{r}\}$ for the independent particle model, it will evidently take some time before we can hope to obtain a complete evaluation of the solutions for the coupled equations for $\varphi^0_{k\alpha}\{\mathbf{r}\}$ and $\varphi^1_{k\alpha}\{\mathbf{r}\}$ using an accurate estimate of the coupling coefficient $\Delta\{\mathbf{r}\}$. In the meanwhile, as an immediately available approximation, offering the best that can be hoped for as a provisional estimate in the short run, we can use an ansatz of the standard BCS kind, which means adopting the prescription

$$U_{k\alpha,l\beta} = \cos\theta_{k\alpha}\delta_{kl}\delta_{\alpha\beta}, \qquad V_{k\alpha,l\beta} = \sin\theta_{k\alpha}\delta_{-k,l}\delta_{\alpha\beta}. \tag{80}$$

Comparing with (53) and (54), the Bogoliubov particle–hole doublet reduces to

$$\varphi^0_{k\alpha}\{\mathbf{r}\} = \varphi_{k\alpha}\{\mathbf{r}\}\cos\theta_{k\alpha}, \qquad \varphi^1_{k\alpha}\{\mathbf{r}\} = \varphi_{k\alpha}\{\mathbf{r}\}\sin\theta_{k\alpha}, \tag{81}$$

where the single component wave functions $\varphi_{k\alpha}\{\mathbf{r}\}$ are the (more easily obtainable) independent particle eigenfunctions, which can be seen from the preceding work to be specifiable as solutions of the simple Schroedinger type equation

$$\mathcal{H}'_{\mathrm{ind}}\varphi_{k\alpha} = \mathcal{E}'_{k\alpha}\varphi_{k\alpha}, \tag{82}$$

where, in the static case under consideration at this stage, we simply have

$$\mathcal{E}'_{k\alpha} = \mathcal{E}_{k\alpha} - \mu, \tag{83}$$

where $\mathcal{E}_{k\alpha}$ is the ordinary Bloch energy value as introduced in (19).

It can be seen that the ansatz (81) will provide an exact solution in the limit for which the relevant coupling field matrix elements

$$\Delta_{k\alpha,l\beta} = \int \mathrm{d}^3r\, \varphi^*_{k\alpha}\{\mathbf{r}\}\Delta\{\mathbf{r}\}\varphi_{l\beta}\{\mathbf{r}\}, \tag{84}$$

reduce to diagonal form, so that we have

$$\Delta_{k\alpha,l\beta} = \Delta_{k\alpha}\delta_{kl}\delta_{\alpha\beta}, \tag{85}$$

using the notation

$$\Delta_{k\alpha} = \Delta_{k\alpha,k\alpha}. \tag{86}$$

The relation (86) will be a good approximation when $\Delta_{k\alpha}$ remains close to a fixed value Δ_{F} (which can be taken without loss of generality to be real and positive by choosing the relevant phase) in the neighbourhood of the Fermi surface, and it will evidently hold exactly when the coupling constant is uniform, so that $\Delta\{\mathbf{r}\} = \Delta_{\mathrm{F}} = \Delta_{k\alpha}$. In the general case, as a result of the periodicity of $\Delta\{\mathbf{r}\}$ the pairing field matrix elements will automatically be diagonal in phase space, namely $\Delta_{k\alpha,l\beta} = \delta_{kl}\Delta_{k\alpha,k\beta}$. However the pairing interactions may couple single particle states belonging to different bands and it will only be an approximation to neglect those contributions when, for instance, $\Delta\{\mathbf{r}\}$ is a field of the radially dependent form that has been obtained [14] within the Wigner–Seitz approximation. Actually the only nonvanishing matrix elements are those relating independent single particles states belonging to the same irreducible representation of the space group [22], which means that only the band states having the same symmetry properties may be coupled. Subject to the validity of (85), the BCS ansatz (81) will reduce the Bogoliubov system of differential equations (62) to a purely algebraic eigenvalue system whose solutions have the well-known form

$$\text{€}_{k\alpha} = \sqrt{\mathcal{E}'^{2}_{k\alpha} + \Delta^2_{k\alpha}}, \tag{87}$$

$$\cos^2\theta_{k\alpha} = \frac{\text{€}_{k\alpha} + \mathcal{E}'_{k\alpha}}{2\text{€}_{k\alpha}}, \qquad \sin^2\theta_{k\alpha} = \frac{\text{€}_{k\alpha} - \mathcal{E}'_{k\alpha}}{2\text{€}_{k\alpha}}. \tag{88}$$

It can be seen that the ansatz (81) has the effect of reducing the Bogoliubov transformation to the simple form given by

$$\hat{c}_{-\sigma k\alpha} = \cos\theta_{k\alpha}\hat{\gamma}_{-\sigma k\alpha} + \sigma\sin\theta_{k\alpha}\hat{\gamma}^{\dagger}_{\sigma-k\alpha}, \tag{89}$$

which is equivalent to taking

$$\hat{\gamma}_{-\sigma k\alpha} = \cos\theta_{k\alpha}\hat{c}_{-\sigma k\alpha} - \sigma\sin\theta_{k\alpha}\hat{c}^{\dagger}_{\sigma-k\alpha}. \tag{90}$$

It follows from this that for the state $|_{\{\mu\}}\rangle$ characterised by (71), the expectation values of the Bloch wave vector dependent number density operators $\hat{n}_{\sigma k\alpha}$ introduced in (26) will be given by

$$\langle_{\{\mu\}}|\hat{n}_{\sigma k\alpha}|_{\{\mu\}}\rangle = \sin^2\theta_{k\alpha}. \tag{91}$$

This result is interpretable as expressing the effect commonly described as a smearing of the Fermi surface, whereby the smoothed out Bloch wave vector space distribution (91) replaces the hard cutoff expressed by the Heaviside formula (49) that applies in limit when the pairing interaction is ignored.

7. Formula for the mobility tensor

When the static contribution characterised by (61) is extended by the inclusion of the current constraint term proportional to the momentum covector $p_i = \hbar q_i$ in the effective energy (34), it can be seen from (15) that as in the independent particle limit, its effect to first order will be entirely taken into account by merely making the gauge adjustment $a_i = -q_i$, in the kinetic energy operator (10), which means changing k_i to $k_i - q_i$ in

Eq. (63). The first order effect of the current will therefore be given, according to (44), by the adjustment

$$\mathcal{E}'_{k\alpha} \mapsto \mathcal{E}'_{\{p\}k\alpha} = \mathcal{E}'_{(k-q)\alpha}, \tag{92}$$

for the single particle energy, and by the ensuing set of infinitesimal transformations

$$\hat{\gamma}_{\sigma k\alpha} \mapsto \hat{\gamma}_{\{p\}\sigma k\alpha}, \qquad |_{\{\mu\}}\rangle \mapsto |_{\{\mu,p\}}\rangle, \tag{93}$$

of quasiparticle operators and state vector, while particularly, in the framework of the BCS approximation based on the neglect of interband couplings, the Bogoliubov angles introduced in (81) will undergo a corresponding adjustment

$$\theta_{k\alpha} \mapsto \theta_{\{p\}k\alpha}. \tag{94}$$

As in the absence of pairing, in the strict BCS case characterised by a fixed gap value, $\Delta_{k\alpha} = \Delta_{\rm F}$, the result will still be describable just as a uniform displacement $\delta k_i = q_i$ in the space of Bloch wavevectors k_i.

As the adjusted version of (30), it can be seen that for any state $|\rangle$ satisfying the simplicity condition that except for the diagonal contributions characterised by $\sigma' = \sigma$, $l_i = k_i$ and $\alpha = \beta$ the contributions of the expectation values $\langle|\hat{c}^{\dagger}_{\{p\}\sigma k\alpha}\hat{c}_{\{p\}\sigma' l\beta}|\rangle$ will vanish—or be negligible to the order of approximation under consideration—the mean current defined by (47) will be given for each spin value by the formula

$$\bar{n}^i_\sigma = \sum_{k,\alpha} v^i_{k\alpha}\langle|\hat{n}_{\{p\}\sigma k\alpha}|\rangle. \tag{95}$$

In the framework of the BCS approximation this formula will be applicable, in particular, to the conducting reference state $|\rangle = |_{\{\mu,p\}}\rangle$, so that by the adjusted analogue of (89) the ensuing replacement of the formula (48), for the mean current in this state, will be obtainable from the substitution

$$\langle_{\{\mu,p\}}|\hat{n}_{\{p\}\sigma k\alpha}|_{\{\mu,p\}}\rangle = \sin^2\theta_{\{p\}k\alpha}, \tag{96}$$

which leads to the expression

$$\bar{n}^i = \sum_\sigma \bar{n}^i_\sigma = 2\sum_{k,\alpha} v^i_{k\alpha}\sin^2\theta_{\{p\}k\alpha} \tag{97}$$

for the corresponding total current.

Since the total current evidently cancels out in the unperturbed static state $|_{\{\mu\}}\rangle$, the quantity given by (97) will be expressible to first order, in the weak current limit with which we are working, as

$$\bar{n}^i = 2\sum_{k,\alpha} v^i_{k\alpha} p_j \frac{\partial(\sin^2\theta_{\{p\}k\alpha})}{\partial p_j}. \tag{98}$$

The conclusion to be drawn from this is that the value of the current will be given to linear order by an expression of the same general form

$$\bar{n}^i = p_j\mathcal{K}^{ij}, \tag{99}$$

as in the absence of pairing, but with required mobility tensor now given by an expression of the form

$$\mathcal{K}^{ij} = 2\sum_{k,\alpha} v^i_{k\alpha} \frac{\partial(\sin^2\theta_{\{p\}k\alpha})}{\partial p_j}. \tag{100}$$

It follows from (92) that in this small $|p|$ limit we shall have

$$\frac{\partial \mathcal{E}'_{\{p\}k\alpha}}{\partial p_i} = -\frac{\partial \mathcal{E}'_{\{p\}k\alpha}}{\hbar \partial k_i} = -\frac{\partial \mathcal{E}_{k\alpha}}{\hbar \partial k_i} = -v^i_{k\alpha}, \tag{101}$$

and hence that the partial derivative in (98) can be evaluated in the BCS approximation as

$$\frac{\partial(\sin^2\theta_{\{p\}k\alpha})}{\partial p_i} = -v^i_{k\alpha}\frac{\partial(\sin^2\theta_{k\alpha})}{\partial \mathcal{E}'_{k\alpha}} = -v^i_{k\alpha}\frac{\partial(\sin^2\theta_{k\alpha})}{\partial \mathcal{E}_{k\alpha}}, \tag{102}$$

in which $\sin^2\theta_{k\alpha}$ is given as a function of the quantity $\mathcal{E}'_{k\alpha} = \mathcal{E}_{k\alpha} - \mu$ and of Δ_{F} by (88). The mobility tensor will therefore be expressible as

$$\mathcal{K}^{ij} = -2\sum_{k,\alpha} \frac{\partial(\sin^2\theta_{k\alpha})}{\partial \mathcal{E}_{k\alpha}} v^i_{k\alpha} v^j_{k\alpha}, \tag{103}$$

in which, by (88), the relevant coefficient will be given by

$$\frac{\partial(\sin^2\theta_{k\alpha})}{\partial \mathcal{E}_{k\alpha}} = -\frac{\Delta^2_{\mathrm{F}}}{2\epsilon^3_{k\alpha}}. \tag{104}$$

The translation of the discrete summation formula (103) into the language of continuous integration (in the limit in which the size of the mesoscopic cell is much larger than the lattice spacing) is given by (4).

Except near the base of the neutron star crust where the nuclei may acquire exotic (e.g., “spaghetti” or “lasagna” type) configurations, it is to be expected that the mobility tensor will have the isotropic form

$$\mathcal{K}^{ij} = \mathcal{K}\gamma^{ij}, \tag{105}$$

where

$$\mathcal{K} = \tfrac{1}{3}\gamma_{ij}\mathcal{K}^{ij} = -\frac{2}{3}\sum_{k,\alpha}\frac{\partial(\sin^2\theta_{k\alpha})}{\partial \mathcal{E}_{k\alpha}} v^2_{k\alpha}, \quad v^2_{k\alpha} = \gamma_{ij} v^i_{k\alpha} v^j_{k\alpha}. \tag{106}$$

It is to be observed that subject to the BCS approximation of uniform coupling, meaning that there is a constant gap parameter, $\Delta_{k\alpha} = \Delta_{\mathrm{F}}$, the formulae (103) and (106) will be convertible, using integration by parts, to the form

$$\mathcal{K}^{ij} = 2\sum_{k,\alpha}\frac{\sin^2\theta_{k\alpha}}{\hbar^2}\frac{\partial^2\mathcal{E}_{k\alpha}}{\partial k_i \partial k_j}, \qquad \mathcal{K} = \frac{2}{3}\sum_{k,\alpha}\frac{\sin^2\theta_{k\alpha}}{\hbar^2}\gamma_{ij}\frac{\partial^2\mathcal{E}_{k\alpha}}{\partial k_i \partial k_j}. \tag{107}$$

This latter formula is useful for the evaluation of the corresponding effective mass $m_\star$ as defined by

$$m_\star = n/\mathcal{K}, \tag{108}$$

in terms of the relevant total particle number density as given by the prescription

$$n = \sum_{\sigma} \langle |\hat{\hat{n}}_{\sigma}|\rangle = 2 \sum_{k,\alpha} \sin^2 \theta_{k\alpha}, \tag{109}$$

in which, if we only wish to count unbound neutrons, the summation should be taken only for values above a lower cutoff below which the states are bound so that the corresponding values of the velocity $v_{k\alpha}$ will vanish.

The concept of an effective mass has traditionally been a source of confusion as different definitions have been used in different contexts. Moreover in solid state physics one is often more interested in electric charge (not mass) whose transport is related to the electric field E_i via an Ohm type law as

$$j^i = en^i = \sigma^{ij} \mathrm{E}_j, \tag{110}$$

where n^i is the electron current density, and σ^{ij} is the relevant electric conductivity tensor. While this conductivity tensor σ^{ij} has the advantage of relating macroscopic measurable quantities, it depends on the dynamical evolution of the medium unlike the newly introduced mobility tensor $\mathcal{K}^{ij}$, on which the effective mass $m_\star$ is defined. The electric conductivity tensor will be given by an expression of the form

$$\sigma^{ij} = e^2 \tau \mathcal{K}^{ij} \tag{111}$$

in which τ is a timescale characterising the rate of decay (by various scattering processes) towards the zero current state that is the only locally stable configuration in the "normal" case. The kind of superconducting case with which we are concerned may be described as a limit in which the relevant timescale τ is infinite, so that the conductivity σ^{ij} will also be infinite, even though the mobility tensor $\mathcal{K}^{ij}$ has a well behaved finite value as in the "normal" case. However it is important to understand that the reason why the relevant timescale τ is effectively infinite in the superconducting case is *not* because relevant scattering cross sections are small (as in the case of a "normal" good conductor) but rather because the current carrying configuration is locally *stable* (with respect to scattering processes that may be quite strong) in a superconducting state, for reasons that will be reviewed in the next section.

It is to be remarked that the formula for the mobility tensor (103) is very similar to the formula obtained without pairing correlations, the Heaviside unit step distribution being merely smeared. In particular the same velocities appear in these formulae. One might have naively guessed that apart from the particle state distribution which is smoothed, the relevant velocity would have been given not by the ordinary group velocity $v^i_{k\alpha}$ given as the momentum space gradient of the energy distribution $\mathcal{E}_{k\alpha}$ by (2) but by the analogously defined quantity $\tilde{v}^i_{k\alpha}$ obtained by substituting $\epsilon_{k\alpha}$ in place of $\mathcal{E}_{k\alpha}$, namely

$$\tilde{v}^i_{k\alpha} = \frac{1}{\hbar} \frac{\partial \epsilon_{k\alpha}}{\partial k_i}. \tag{112}$$

Actually this latter "pseudovelocity" is interpretable as a mean velocity between particles and holes, since $\epsilon_{k\alpha}$ is the energy of a quasiparticle which is a mixture of particles and holes. More specifically, when (as in the simple BCS case for an homogeneous system) the gap parameter is independent of the momentum, this modified velocity will be given by the expression

$$\tilde{v}^i_{k\alpha} = v^i_{k\alpha} \frac{\mathcal{E}'_{k\alpha}}{\epsilon_{k\alpha}}, \tag{113}$$

from which it can be seen that $\tilde{v}^i_{k\alpha}$ will vanish at the Fermi surface characterised by $\mathcal{E}_{k\alpha} = \mu$, where the number of particles is equal to the number of holes.

In the limit for which, in so far as the unbound neutrons are concerned, the effect of the crustal nuclei is small (either because the nuclei occupy only a small part of the volume, as will be the case just above the neutron drip transition, or because the nuclear surface is very diffuse, as will be the case near the base of the crust) we shall have

$$\frac{1}{\hbar^2} \frac{\partial^2 \mathcal{E}_{k\alpha}}{\partial k_i \partial k_j} = \frac{1}{m^{\oplus}} \gamma_{ij}, \tag{114}$$

where $m^{\oplus}$ is the uniform mass scale appearing in the kinetic energy operator, which will be comparable with, but for precision somewhat less that, the ordinary neutron mass m. It can be seen by comparing (107) and (109) that in this approximately uniform limit the effective mass for the unbound neutrons will be given simply by $m_\star = m^{\oplus}$, regardless of whatever the value of the gap parameter Δ may be.

8. Superconductivity property and critical current

An unsatisfactory feature of the rather profuse contemporary literature dealing with various kinds of what is commonly referred to as "superconductivity" in astrophysically relevant contexts (including such exotic varieties as colour superconductivity in quark condensates) is the rarity of any serious theoretical consideration of the actual property of superconductivity in the technical sense, meaning the possibility of having a relatively moving current that is effectively stable, or in stricter terminology *metastable*, with respect to small perturbations—such as would normally give rise to a dissipative damping mechanism of a resistive or viscous kind.

In the astrophysical literature concerned with pulsars it has generally been taken for granted that neutron currents of the kind considered in the present work actually are characterised by superconductivity in the sense of being metastable with respect to relevant kinds of perturbation. In this section we shall investigate the conditions under which this supposition of metastability is indeed valid. The issue is that of the stability, for small but finite values of the momentum covector p_i, of the superconducting reference state $|\rangle = |_{\{\mu,p\}}\rangle$ that is characterised by the minimisation of the combination (34).

The conducting state $|_{\{\mu,p\}}\rangle$ was derived by minimising the energy expectation $\langle|\hat{H}|\rangle$ subject to the condition that the particle number expectation $\langle|\hat{\hat{n}}|\rangle$ and the current expectation $\langle|\hat{\hat{n}}^i|\rangle$ were held fixed. It is physically reasonable to suppose the particle number expectation $\langle|\hat{\hat{n}}|\rangle$ really will be preserved under the conditions of chemical equilibrium that are envisaged in the relevant applications, but no such consideration applies to the current expectation $\langle|\hat{\hat{n}}^i|\rangle$ which in a "normal" state would tend to be damped down by many conceivable kinds of scattering process. The physically pertinent question is therefore whether

$|_{\{\mu,p\}}\rangle$ will still minimise $\langle|\hat{H}|\rangle$ with respect to small relevant perturbations—subject of course to the preservation of the particle number expectation $\langle|\hat{\hat{n}}|\rangle$ as before—when the prior assumption of preservation of $\langle|\hat{\hat{n}}^i|\rangle$ is abandoned. Subject to the particle number conservation condition

$$\delta\langle|\hat{\hat{n}}|\rangle = 0, \tag{115}$$

this stability requirement is equivalent to the condition of minimisation of $\langle|\hat{H}'|\rangle$ meaning that any admissible perturbation must satisfy

$$\delta\langle|\hat{H}'|\rangle > 0, \tag{116}$$

where according to the notation introduced in (35),

$$\hat{H}' = \hat{H}'_{\{p\}} + p_i \hat{\hat{n}}^i. \tag{117}$$

According to the reasoning of the previous section, the relevant adjustment of (70) will give us

$$\hat{H}'_{\{p\}} = \sum_{\sigma,k,\alpha} \text{€}_{\{p\}k\alpha}\big(\hat{\gamma}^{\dagger}_{\{p\}\sigma k\alpha}\hat{\gamma}_{\{p\}\sigma k\alpha} - \sin^2\theta_{\{p\}k\alpha}\big), \tag{118}$$

so that the specifications (117) and (95) provide the variation formula

$$\delta\langle|\hat{H}'|\rangle = \sum_{\sigma,k,\alpha} \big(\text{€}_{\{p\}k\alpha}\delta\langle|\hat{\gamma}^{\dagger}_{\{p\}\sigma k\alpha}\hat{\gamma}_{\{p\}\sigma k\alpha}|\rangle + p_i v^i_{k\alpha}\delta\langle|\hat{n}_{\{p\}\sigma k\alpha}|\rangle\big), \tag{119}$$

in which, for the BCS case, it can be seen from (87) that we shall have

$$\text{€}_{\{p\}k\alpha} = \sqrt{\mathcal{E}'^2_{\{p\}k\alpha} + \Delta^2_{k\alpha}}. \tag{120}$$

In this BCS case, the action on the conducting state $|_{\{\mu,p\}}\rangle$ of a typical quasiparticle creation operator $\hat{\gamma}^{\dagger}_{\{p\}\uparrow k\alpha}$ will provide only three nonvanishing terms in the sum (119), namely those given by

$$\delta\langle|\hat{\gamma}^{\dagger}_{\{p\}\uparrow k\alpha}\hat{\gamma}_{\{p\}\uparrow k\alpha}|\rangle = 1, \tag{121}$$

together with the number variation contributions

$$\delta\langle|\hat{n}_{\{p\}\uparrow k\alpha}|\rangle = \cos^2\theta_{\{p\}k\alpha}, \qquad \delta\langle|\hat{n}_{\{p\}\downarrow -k\alpha}|\rangle = -\sin^2\theta_{\{p\}-k\alpha}. \tag{122}$$

It follows from the symmetry properties $v^i_{k\alpha} = -v^i_{-k\alpha}$ and $\theta_{k\alpha} = \theta_{-k\alpha}$ that the explicit dependence on the Bogoliubov angle will cancel out at first order in the net energy contribution provided in (119) by such an excitation, so that this energy contribution will be positive if and only if

$$\text{€}_{\{p\}k\alpha} + p_i v^i_{k\alpha} > 0. \tag{123}$$

It is to be noted that such an individual quasiparticle excitation will in general violate the requirement (115) to the effect that the number of real particles should be conserved, but it is evident from (122) that such violations may have either sign and so can be cancelled out by the combined effect of two or more elementary excitations. What, in a stable case, can not be cancelled out is the combined effect of several quasiparticle energy contributions in (119): it can be seen that the quasiparticle energy contributions will always add up to give the positive result needed for stability provided the inequality (123) is satisfied for all admissible modes.

The stability condition (123) that we have derived for a BCS type inhomogeneous superconductor is consistent with Landau's classical treatment based on vaguer heuristic arguments in the context of superfluid Helium 4 [24]. Since the quasiparticle energy $\in_{\{p\}k\alpha}$ is always positive, it is clear that the stability condition (123) will always be satisfied if $p_i v^i_{k\alpha} > 0$. Therefore an instability in the superfluid conducting state can only occur when $p_i v^i_{k\alpha} < 0$. Rewriting the inequality (123) as $\in_{\{p\}k\alpha} > -p_i v^i_{k\alpha}$, squaring both sides and substituting the expression (43) for $\mathcal{E}'_{\{p\}k\alpha}$ in (120) we see that in the BCS case there is a remarkable simplification (which does not seem to have been pointed out before) whereby the terms that are nonlinearly dependent on the momentum covector p_i cancel out, so that the superfluid conducting state will be stable if the following inequality is satisfied

$$2 p_i v^i_{k\alpha} \mathcal{E}'_{k\alpha} < \in^2_{k\alpha}. \tag{124}$$

This can be rewritten in terms of the "pseudovelocity" introduced in (112) as

$$p_i \tilde{v}^i_{k\alpha} < \frac{1}{2} \in_{k\alpha}, \tag{125}$$

which simplifies to the following form whenever the BCS gap is independent of the momentum

$$p_i \frac{\partial}{\partial k_i} \left(\ln\{\in^2_{k\alpha}\} \right) < \hbar. \tag{126}$$

The inequality (124) is evidently verified for elementary excitations above the Fermi level for which $\mathcal{E}'_{k\alpha} > 0$. In the derivation of the inequality (124) we have assumed that $p_i v^i_{k\alpha} < 0$. Actually there will always be some modes for which this is satisfied since whenever $p_i v^i_{k\alpha} > 0$ we shall have $p_i v^i_{-k\alpha} < 0$.

The stability condition (123) (which has to be satisfied for all modes) can therefore be restated as the requirement that the magnitude p of the mean particle momentum covector p_i lies below some critical threshold $p_{\rm c}$

$$p < p_{\rm c} \tag{127}$$

in which, for an approximately isotropic distribution depending only on the magnitude k of the wavenumber covector k_i, the critical value $p_{\rm c}$ will be given by

$$p_{\rm c} \approx \min_{k,\alpha} \left\{ \frac{1}{2 v_{k\alpha}} \left(|\mathcal{E}'_{k\alpha}| + \frac{\Delta^2_{k\alpha}}{|\mathcal{E}'_{k\alpha}|} \right) \right\}. \tag{128}$$

Since $\mathcal{E}'_{k\alpha}$ vanishes on the Fermi surface, it is clear from (128) that $p_{\rm c}$ will also vanish—so that there will be no phenomenon of superconductivity—not only when the gap $\Delta_{k\alpha}$ vanishes everywhere, but even when it vanishes just in the neighbourhood of the Fermi surface. When the mean value $\Delta_{\rm F}$ of the gap at the Fermi surface is nonzero but small compared with the other relevant energy scales—as will typically be the case—it can be seen that the minimum in (128) will be attained for energy values differing from the Fermi value by a small but finite positive or negative amount that will be given approximately by $\mathcal{E}'_{k\alpha} \approx \pm\Delta_{\rm F}$. In such a case, it follows that the critical momentum value (128) will be expressible in terms of the mean value $v_{\rm F}$ of the group velocity magnitude $v_{k\alpha}$ at the Fermi surface by the approximation

$$p_{\rm c} \approx \frac{\Delta_{\rm F}}{v_{\rm F}}. \tag{129}$$

Introducing the critical velocity as $v_{\rm c} \equiv p_{\rm c}/m_\star$, the criterion (129) can be written as

$$v_{\rm c} \approx \frac{\Delta_{\rm F}}{m_\star v_{\rm F}}. \tag{130}$$

In the limit of an homogeneous medium for which $m_\star = m^\oplus$, the critical velocity reduces to an expression which is commonly found in the literature concerning homogeneous electron superconductivity in metals [25], namely

$$v_{\rm c}^{(0)} \approx \frac{\Delta_{\rm F}^{(0)}}{\hbar k_{\rm F}}, \tag{131}$$

(using the superscript $^{(0)}$ to indicate what would be obtained for uniform values of the microscopic effective mass $m^\oplus$ and potential V) where $k_{\rm F}$ is the radius of the Fermi sphere. It must be emphasized however that the critical velocity of an inhomogeneous superfluid (such as the neutron superfluid in the inner layers of a neutron star crust) will differ from the estimate (131) by a factor

$$\frac{v_{\rm c}}{v_{\rm c}^{(0)}} = \frac{\Delta_{\rm F}}{\Delta_{\rm F}^{(0)}} \frac{S_{\rm F}}{S_{\rm F}^{(0)}}, \tag{132}$$

where $S_{\rm F}$ and $S_{\rm F}^{(0)}$ are the Fermi surface areas in the inhomogeneous and homogeneous cases, respectively. Since the opening of band gaps in the single particle energy spectrum decreases the Fermi surface area, the critical velocity is therefore expected to be smaller than the expression (131) assuming that $\Delta_{\rm F} \approx \Delta_{\rm F}^{(0)}$.

For a gap of the order of an MeV, in a region where the kinetic contribution to the Fermi energy has a typical value of the order of a few tens of MeV, the formula (129) implies a critical momentum value corresponding to a kinetic energy of relative motion of the order of hundreds of keV per neutron. This is comparable with the total kinetic energy of rotation in the most rapidly rotating pulsars. However the relative rotation speeds of the neutron currents that are believed to be involved in pulsar glitch phenomena are very much smaller—by factors of 10^{-3} or even far less [23]—than the absolute rotation speeds of the neutron star. In all such cases it may therefore be concluded that the superfluidity criterion (129) will be satisfied within an enormous confidence margin.

It is to be remarked that in the more thoroughly investigated context of laboratory superfluidity [24] Landau's simple linear formulation of the stability problem in terms just of phonons provides only an upper limit on the critical momentum whose true value is considerably reduced by the less mathematically tractable—since essentially nonlinear—effect of protons. Analogous considerations presumably apply in the present context. This means that although our present treatment places the estimate (129) on a sounder footing than was provided by any previous work of which we are aware, it should still be considered just as an upper bound on the true critical value which is likely to be substantially reduced by non-linear effects whose mathematical treatment is beyond the scope of the methods used here. Despite this caveat, the prediction of genuine superconductivity in the context of glitching neutron star crusts should be considered to be very robust. The justification for such confidence is that—according to the considerations outlined in the preceding paragraph—the relevant magnitudes of the neutron currents in question correspond to values of the neutron momentum p that are extremely small compared with the order of magnitude given by (129). For such very low amplitude currents there is no obvious reason to doubt the validity of conclusions—including estimates of effective masses, as well as the prediction of genuine superfluidity—that are based on the simple kind of linearised treatment used here.

9. Conclusions

In the middle layers of the crust, where the effect of inhomogeneities will be important, our previous analysis neglecting the effect of the superfluid gap lead to the prediction [6] that there will be a strong “entrainment” effect whereby the value of $m_\star$ will become very large compared with m. This prediction has now been confirmed by an analysis [7] based on the phenomenological model of Oyamatsu et al. [26] where values as large as $m_\star/m \sim 15$ has been found at a baryon density $n_b = 0.03\ \text{fm}^{-3}$. Our present analysis indicates that this conclusion will not be significantly affected by taking the relevant pairing gap Δ into account. Thus although the pairing is essential for the actual phenomenon of superconductivity, on the other hand, in so far as the effective mass is concerned, neglect of the effect of superconductivity pairing will indeed be justifiable as a robust first approximation, at least for moderate values of the gap parameter compared to the kinetic contribution to the Fermi energy.

The unimportance of pairing from the point of view of entrainment, which has been usually assumed (see for instance Borumand et al. [3]) and explicitly shown in the present work, can be seen from the consideration that, when $\Delta_{\alpha k}$ is small compared to the Fermi energy μ, the coefficient (104) will be very small everywhere except in a thin layer with width of the order of $\Delta_{\alpha k}$ near the Fermi surface locus where $\mathcal{E}_{\alpha k} = \mu$, which means that when the coupling is weak its effect will be entirely negligible. In sensitive cases for which the geometry of the energy contours near the Fermi surface is complicated by band effects, a moderately strong pairing effect might make a significant difference by smoothing out variations of the mobility tensor as a function of density, but does not seem likely that this smearing effect would make much difference to the large scale average properties of the mobility tensor. In other words the effective mass is expected to be much more sensitive to band gaps than to the pairing gap. The main reason is that Δ appears only in the number density distribution whereas band gaps (resulting from Bragg scattering of dripped neutrons by crustal nuclei) have a strong influence upon the neutron velocity $v^i_{\alpha k}$ which is vanishingly small in this case.

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