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COVARIANT ANALYSIS OF NEWTONIAN MULTI-FLUID MODELS FOR NEUTRON STARS: I MILNE–CARTAN STRUCTURE AND VARIATIONAL FORMULATION

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This is the first of a series of articles showing how 4 dimensionally covariant analytical procedures developed in the context of General Relativity can be usefully adapted for application in a purely Newtonian framework where they provide physical insights (e.g. concerning helicity currents) that are not so easy to obtain by the traditional approach based on a $3 + 1$ spacetime decomposition. After an introductory presentation of the relevant Milne spacetime structure and the associated Cartan connection, the essential principles are illustrated by application to the variational formulation of simple barotropic perfect fluid models. This variational treatment is then extended to conservative multiconstituent self-gravitating fluid models of the more general kind that is needed for treating the effects of superfluidity in neutron stars.

1. Introduction

As a generalization of previous work[1] on the special case of Landau's two-constituent superfluid model, using a 4-dimensionally covariant treatment of the kind pioneered by Peradzynski,[2] this article presents a coherent fully covariant approach to the construction and application of Newtonian fluid models of the more general kind required in the context of neutron star phenomena in cases for which it is necessary to allow for independent motion of neutronic and protonic constituents.

Whereas a simple perfect fluid model is sufficient for deriving the most basic features of neutron stars (such as the radius for a given mass and the oblateness for a given angular momentum) models involving at least two independent constituents (of which at least one is superfluid) are needed to account for the details revealed by pulsar frequency observations. If quantitative accuracy is needed, the high mean densities of neutron stars require the use of a general relativistic treatment. In accordance with this requirement, in applications for which a simple perfect fluid model is sufficient, use of fully relativistic models has been standard practice since the outset of neutron star theory. However, when more elaborate models have been needed, most work has relied on less accurate Newtonian models, either because the relevant relativistic models had not been developed or because, even if available in principle, the relevant relativistic models were too difficult to apply in practice.

When both non-relativistic and relativistic versions are available, as is the case[1,3] for multiconstituent superfluid models, the question of which is most appropriate for a given purpose depends not just on considerations of intrinsic accuracy or computational economy but also on questions of extrinsic compatibility with the relevant background framework. Thus for treating perturbations of a zeroth order global configuration described by a fully relativistic perfect fluid model, what will usually be most convenient is the employment[4,5] of a two constituent fluid model that is also fully relativistic. However to deal with interactions with a solid crust described by a Newtonian elasticity model (since although appropriate relativistic elasticity models are available in principle,[3] their technical complexity has so far prevented them from being effectively applied in practice) it may be more practical[6,7] to use a two constituent fluid model that is also non-relativistic.

The purpose of this article is to show how to set up and apply a *fully covariant* formulation of the kinds of non-relativistic multiconstituent fluid dynamical models that are needed for such cases. Using a more traditional kind of formulation based on a preferred space reference frame (which complicates the treatment of effects such as helicity conservation, but facilitates the generalization to allow for electromagnetism) a complementary development of the same class of Newtonian models has recently been provided by Prix.[8]

The previous analysis, on which the present work is based,[1] was restricted to the Landau model which involves just a single massive particle constituent together with a second constituent, representing entropy, that is massless in the Newtonian limit. The more general analysis presented here covers cases (including the historic prototype of the Andreev–Bashkin model[9] for a superfluid helium mixture) involving at least two independent constituents representing particles of which *both* kinds are massive, as in the particularly relevant example of the application to independently moving superfluid neutrons and protons. One of the side issues that will be dealt with here is the relationship between the effect of entrainment (whereby the relevant momenta deviate from the corresponding particle velocities) and the "effective masses" that have been defined in different ways in the published literature.

As has already been pointed out by Peradzynski,[2] a noteworthy advantage (even for non-relativistic description) of using a 4-dimensionally covariant treatment, like that of the "canonical" approach developped here, is the possibility of exploiting Cartan type methods of mathematical analysis, involving the use of antisymmetric differential forms for the construction, not just of vorticities, but also (as in the relativistic case)[10] of a more elaborate category of helicity currents. The technical intricacy of the construction procedure for these helicity currents is such that they would be very awkward to deal with (and are therefore usually ignored) in the conventional kind of approach based on a non-covariant frame based formulation of Newtonian theory.

When setting up a covariant description of Newtonian theory it is worthwhile to recall how, having already jettisoned the distinction between time and space in his special relativity theory, Einstein went on to jettison the distinction between spacetime and gravitation in his general relativity theory: in the latter theory the specification of the spacetime metric $g_{\mu\nu}$ automatically includes the specification of gravity via the corresponding Riemannian covariant differentiation operator ∇_μ which is determined by a connection with components $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ that are given by the well known Christoffel formula

$$\Gamma_{\mu}{}^{\nu}{}_{\rho} = g^{\nu\sigma}\left(g_{\sigma(\mu,\rho)} - \frac{1}{2}g_{\mu\rho,\sigma}\right),$$

using a comma for partial differentiation with respect to the (arbitrarily chosen) spacetime coordinates x^μ, and round brackets for index symmetrisation. However a distinction between spacetime and gravity can be made in the Newtonian analogue of Einstein's Riemannian structure, namely something that is known[11,12] as a Newton–Cartan structure — whose not so widely familiar principles will be recapitulated below — involving a non-Riemannian covariant differentiation operator D_μ that is determined by a Cartan connection with components $\omega_{\mu}{}^{\nu}{}_{\rho}$ that are *not* obtainable from a Christoffel formula because the Newtonian structure does not specify the non-degenerate spacetime metric that would be needed. In contrast with the inextricable case of general relativity, it is possible in the Newtonian case to extricate the specification of the gravitational field, as given by the (non-tensorial) Cartan connection components $\omega_{\mu}{}^{\nu}{}_{\rho}$, from that of the underlying spacetime manifold. Prior to any knowledge of the gravitational field (i.e. the Cartan connection) the only endowment of the underlying Newtonian spacetime manifold consists just of what is describable as a Milne structure, a concept that will be briefly recapitulated in the immediately following section.

Like the Minkowski structure of special relativity theory, the Milne structure of Newtonian spacetime does not involve any free parameters or fields, i.e. it is intrinsically unique (modulo diffeomorphisms). Nevertheless, despite its intrinsic simplicity the nature of this Newtonian spacetime structure is just sufficiently subtle to have prevented it from being properly understood until Milne's introduction of Newtonian cosmology theory a couple of decades *after* Einstein's introduction of general relativity — though not so long after Friedmann's foundation of the corresponding general relativistic cosmology theory — at about the same time as Cartan's epoch making work on the development of the appropriate mathematical machinery. Milne's breakthrough was based on the extrapolation to a global level of Einstein's earlier observation — originally in a Newtonian context, as a guiding principle for the construction of the corresponding relativistic theory — of the equivalence at a local level between gravitation and acceleration.

2. Covariant Description of Newtonian Spacetime

Whereas a fully covariant treatment is generally recognised to be indispensable for formulating general relativistic models, on the other hand, for their Newtonian analogues, the usual practice is to rely entirely on an "Aristotelian" decomposition whereby spacetime is considered as a direct product of a flat Euclidean 3-space with a one-dimensional Euclidean time line. Any such *Aristotelian structure* will be characterized by a corresponding class of Aristotelian coordinate systems, which consist of a set of ordinary Cartesian (orthonormal) space coordinates X^i ($i = 1, 2, 3$) together with a Newtonian time coordinate t which is physically well defined modulo a choice of time origin. These coordinate systems are mapped onto each other by the transformations of a 7 parameter Aristotelian symmetry group, consisting of the product of the 6 parameter group of Euclidean translations and rotations with the 1 parameter group of time translations.

Whereas the time coordinate t of such a system is physically well defined (modulo an arbitrary adjustment of the origin) it has been generally recognised since the foundation of Newtonian theory that in a generic application there will be no uniquely preferred Aristotelian structure, but that the theory will be invariant with respect to a group of *gauge transformations* relating different Aristotelian product structures that all share the same constant time sections but that do not have the same sections of constant space position (as measured by fixed values of the Aristotelian space coordinates X^i). Any such gauge transformation will be specifiable by a mapping of one of the (7 parameter family of) sets of Aristotelian coordinates of a particular Aristotelian structure to a set belonging to another such structure. Since the flat constant time sections are preserved, any such transformation must be expressible just as a mapping $X^i \mapsto \breve{X}^i$, $t \mapsto \breve{t}$, for which the time transformation is trivial, $\breve{t} = t$, and for which the new space coordinates $\breve{X}^i$ are given by a time dependent Euclidean transformation. However not all kinds of time dependent Euclidean transformation are admissible. In particular time dependent rotations (belonging to what is known as the Coriolis group) are excluded from the status of gauge transformations because they change the physical comportment of the system (giving rise to what is known as the Coriolis effect).

It turns out that the only admissible gauge transformations between different Aristotelian structures are time dependent space translations as given by a transformation of the form $X^i \mapsto \breve{X}^i$ with

$$\breve{X}^i = X^i - c^i \,, \tag{1}$$

for quantities c^i that are constants in the sense that they have to be independent of the space coordinates X^i, but that are arbitrarily variable as functions of time. For a long time it was generally believed that not all such time dependent translations were admissible, but only the three parameter set of Galilean transformations, meaning those which are *linear* so that the quantities c^i can be taken to be given by expressions of the form $c^i = b^i t$ in which the quantities b^i are constants in the strong sense of being independent not just of the X^i but also of t. A set of Aristotelian structures related by such linear transformations constitutes what may be described as a Galilei structure.

The important point that escaped everybody's notice until, after having been implicitly recognised in Einstein's "equivalence principle," it was finally exploited for the foundation of Newtonian cosmology by Milne[14] is that Newtonian mechanics contains nothing to distinguish any particular preferred Galilei structure. The set of gauge transformations that are admissible in the sense that (unlike generic Coriolis transformations) they preserve the form of the physical laws of motion is not restricted to linear Galilean transformations, but includes generic transformations of the form (1) in which, while independent of the X^i, the quantities c^i are allowed to have an arbitrarily non-linear dependence on the time t. The complete set of all the Aristotelian product decompositions that can be obtained by such Milne gauge transformations constitutes what may be described as the *Milne structure* of spacetime. Thus the (physically unique) Milne structure consists of a family of (physically equivalent) Galilean structures that are related to each other by *accelerated* space translation transformations of the form (1), while each member of the (infinite parameter set) of Galilei structures consists of a (3 parameter) set of Aristotelian structures that are related to each other by *linear* transformations of the form (1).

Although the traditional employment of a particular choice of Aristotelian structure, and in particular of some corresponding set of Aristotelian coordinates $\{t, X^i\}$ is useful for many purposes, such as the exploitation of the flatness and distant parallelism of the preferred (constant time) 3-space sections, the advantages of this are not cost free. The price to be paid includes not just the well known obligation to verify Galilean (and Milne) invariance with respect to changes of the Aristotelian frame. Another, less well known, cost is the loss of access to the elegant and powerful mathematical methods based on tensors, and particularly on Cartan type differential forms, that become available when a fully covariant four-dimensional framework is used.

A convenient feature of the three-dimensional constant time sections in an Aristotelian decomposition is the existence of a physically well defined — and flat — metric having components γ_{ij} that are given simply by the unit matrix with respect to orthonormal Aristotelian space coordinates X^i $(i, j = 1, 2, 3)$. This metric can be used together with its contravariant inverse γ^{ij} for raising and lowering of space indices, including those of the associated antisymmetric volume measure tensor ε_{ijk} (whose non-vanishing components are given by $\pm\sqrt{\det\{\gamma\}}$ with the sign depending on whether the $\{i, j, k\}$ is an even or odd permutation of $\{1, 2, 3\}$) whose contravariant version will be characterized by the normalization condition $\varepsilon^{ijk}\varepsilon_{ijk} = 3!$. A similarly convenient feature of a general relativistic formulation, is the existence of a physically well defined — but not in general flat — spacetime metric giving components $g_{\mu\nu}$ say with respect to coordinates x^μ $(\mu, \nu = 0, 1, 2, 3)$, which determines a corresponding antisymmetric spacetime volume measure tensor with non-vanishing components equal to $\pm\sqrt{-\det\{g\}}$, and that can be used together with its contravariant inverse $g^{\mu\nu}$ for raising and lowering of spacetime indices. Quite apart from the seductive flatness and parallelism properties of the preferred space sections in an Aristotelian decomposition, one of the reasons why fully covariant formulations of Newtonian dynamics are not as widely used as they deserve to be is their lack of an analogous means of raising and lowering of spacetime indices. One of the purposes of this work is to show that, despite this handicap, Newtonian mechanics can nevertheless be set up without too much difficulty in a fully covariant formulation that makes it easy to exploit the technical advantages of freedom to use arbitrarily curved spacetime coordinates x^μ.

The first step is to obtain the fundamental spacetime tensor fields that are available, in lieu of the relativistic metric tensor, for the characterization of Newtonian spacetime. To start with, for any fixed value of the preferred Newtonian time coordinate, t, the embedding mapping $X^i \mapsto x^\mu$ of the corresponding Aristotelian space section will determine physically well defined contravariant spacetime tensor fields with components given by

$$\gamma^{\mu\nu} = \gamma^{ij} x^\mu{}_{,i} x^\nu{}_{,j} \,, \qquad \varepsilon^{\mu\nu\rho} = \varepsilon^{ijk} x^\mu{}_{,i} x^\nu{}_{,j} x^\rho{}_{,k} \,, \tag{2}$$

using a comma to indicate partial differentiation (in this case with respect to the space coordinates X^i). However there is no unambiguously preferred (Galilei invariant) prescription for lowering the indices to obtain corresponding contravariant versions $\gamma_{\mu\nu}$ and $\varepsilon_{\mu\nu\rho}$ because the former is degenerate ($\gamma^{\mu\nu}$ has matrix rank 3 not 4) so it does not have a well defined inverse.

The foregoing consideration means that in Newtonian theory four-dimensional tensor indices will in general have an irrevocably covariant or contravariant nature. The most basic example is that of the preferred covariant unit vector t_μ that is obtained simply as the gradient of the preferred Newtonian time coordinate, i.e. $t_\mu = t_{,\mu}$, and that is a null eigencovector of the degenerate preferred contravariant space metric, and also orthogonal to $\varepsilon^{\mu\nu\rho}$, i.e.

$$\gamma^{\mu\nu} t_\nu = 0 \,, \qquad \varepsilon^{\mu\nu\rho} t_\rho = 0 \,. \tag{3}$$

An exception, having both a covariant and a contravariant version, is that of the antisymmetric spacetime volume measure tensor $\varepsilon_{\mu\nu\rho\sigma}$, which has a physically well defined normalization — despite the lack of a non-degenerate spacetime metric in the Newtonian framework — that is specified by the relation

$$t_\mu = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\nu\rho\sigma} \,, \tag{4}$$

and for which a corresponding contravariant version is unambiguously definable by a normalization condition of the same form,

$$\varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} = -4! \,, \tag{5}$$

as holds in general relativity, which means that it will satisfy

$$\varepsilon^{\mu\nu\rho\sigma} t_\sigma = \varepsilon^{\mu\nu\rho} \,. \tag{6}$$

The fields $\gamma^{\mu\nu}$ and t_μ (the degenerate residue representing all of the algebraic structure that remains from the relativistic spacetime metric in the Newtonian limit) constitute what may be termed a Coriolis structure, since it contains nothing that distinguishes rotating from non-rotating frames. To incorporate this distinction, and thereby complete the covariant specification of Newtonian spacetime, we must consider what to use in place of the Riemannian connection and the associated covariant differentiation operator that in general relativity is uniquely specified by the spacetime metric $g_{\mu\nu}$. In the Newtonian case, a corresponding physically well defined but non-Riemannian connection $\omega_{\mu}{}^{\nu}{}_{\rho}$ and associated covariant differentiation operator D_μ will be provided by the Newton–Cartan structure that is determined by the gravitational field in the manner to be described at the end of the next section. However the only features of the Newtonian spacetime background that are well defined *a priori* (in the absence of information about the gravitational field that specifies the Newton–Cartan structure) are those provided by the Milne structure that was described at the beginning of this section.

The Milne structure by itself (without reference to the gravitational field) does not provide enough information for the unambiguous specification of a connection. What it is capable of providing is a connection that is gauge dependent. Specifically for each choice of gauge, i.e. for each choice of a particular Aristotelian product structure, there will be a corresponding natural connection with components $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ defined by the condition that they simply vanish when evaluated with respect to a corresponding system of Aristotelian coordinates $\{t, X^i\}$ (though not of course if the orthonormal space coordinates X^i were replaced by coordinates of some other, e.g. spherical, kind). This means that with respect to coordinates of this particular type (but not for a system of some other, e.g. spherical, kind) the corresponding covariant differentiation operator ∇_μ will be identifiable with the simple partial differentiation operator ∂_μ.

The gauge dependent connection $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ that is defined in this way has two convenient properties. Firstly, since it is identifiable with partial differentiation in the relevant Aristotelian coordinate system, it is clear that since $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$ the covariant derivative will automatically inherit the analogous commutation property

$$\nabla_\mu\nabla_\nu = \nabla_\nu\nabla_\mu\,, \tag{7}$$

i.e. the connection $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ has the property that (unlike the curved Newton–Cartan connection described below) it is *flat*. The other convenient property is that although (again unlike the gravitational field dependent Newton–Cartan connection $\omega_{\mu}{}^{\nu}{}_{\rho}$) it is gauge dependent, its gauge dependence is of a rather weak kind. Since connection components are unaffected by linear transformations, it follows that $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ will not be affected by gauge transformations of the restricted Galilean type. Choosing a particular Galilean structure (i.e. a linearly related subclass of Aristotelian structures) is equivalent to choosing a particular connection of this flat type. It will be shown below how such a connection is affected by Milne gauge transformations of the more general accelerated type that relate distinct Galilean structures.

According to the preceeding definition, the Milne structure is an equivalence class of Aristotelian (direct product of time and flat space) structures that are related to each other by a gauge group consisting of time dependent space translations. In the modern (post Cartan) technical language of fibre bundle theory this definition can be succinctly reformulated as follows.

In formal mathematical terms, the Milne structure of Newtonian spacetime is formally describable as bundle of three-dimensional Euclidean space fibres (each characterized by its own flat metric γ_{ij}) over a base consisting of a line endowed with a physically preferred time measure (specified by the coordinate t) for which the relevant (Abelian) bundle group consists just of the 3-parameter set of Euclidean space translations (but not rotations), which are expressible in terms of Cartesian coordinate X^i on the fibre simply by transformations of the form (1). (The exclusion of Euclidean rotations from the bundle group is what distinguishes the Milne structure from the more primitive Coriolis structure given just by the specification of the fields t_μ and $\gamma^{\mu\nu}$.)

Any particular section of this Milne bundle, i.e. a representation as a direct product of the base times the fibre, will be interpretable as a particular choice of an Aristotelian structure. In such a direct product structure, the preferred time coordinate t on the base and a choice of Cartesian coordinates X^i on a space section (i.e. coordinates such that metric components γ_{ij} form a unit matrix) will determine a corresponding set of Aristotelian spacetime coordinates x^μ according to the obvious specification $x^0 = t$, $x^i = X^i$. A corresponding connection is thereby definable as the one with respect to which covariant differentiation reduces to partial differentiation, i.e. the one for which $\nabla_\mu = \partial_\mu$ and $\Gamma_\mu{}^\rho{}_\nu = 0$, in these particular coordinates. (This connection could be interpreted as the Riemannian connection provided by a flat Unruh type spacetime metric of the form

$$ds^2 = \gamma_{ij}\, dX^i\, dX^j - C^2 dt^2$$

in which C could be any arbitrarily chosen constant speed, which might be that of light, but which in applications to perturbations in an asymptotically uniform fluid background could more usefully[13] be taken to be the relevant sound speed.)

3. Galilean and Milne Type Gauge Transformations

Although the dynamical equations of ordinary Maxwellian electromagnetic theory are expressible in terms just of gauge independent quantities such as the electric and magnetic fields, it is useful for many purposes, and indispensible for a variational formulation, to employ various gauge dependent entities (starting with the vector potential A_μ). In the present context (of ordinary Newtonian dynamical theory) it is analogously useful for many purposes, and indispensible for a variational formulation, to employ entities whose specification depends on a particular choice of gauge, where in this context a "choice of gauge" is to be interpreted as meaning a particular choice of Aristotelian structure within the large equivalence class of

Aristotelian structures that collectively constitute the Milne structure described above.

In terms of the Aristotelian coordinates $\{t, X^i\}$, the transformation to the analogous coordinates for any alternative bundle section, i.e. any alternative Aristotelian structure, will be expressible by a transformation in which the base coordinate is held fixed, i.e. $t \mapsto t$ while the Cartesian fibre coordinates X^i are transformed according to a relation of the form (1) in which the translation vector c^i is given as an arbitrarily variable function of the base coordinate t. It is evident that the connection specified by the new gauge will be identical with the one specified by the old section, i.e. we shall have $\breve{\Gamma}_{\mu}{}^{\nu}{}_{\rho} = \Gamma_{\mu}{}^{\nu}{}_{\rho}$ and hence $\breve{\nabla}_\mu = \nabla_\mu$, so long as the transformation (1) is *linear*, i.e. so long as the translation is of the Galilean form characterized by the condition that the components

$$b^i = \frac{dc^i}{dt}\,, \qquad b^0 = 0\,, \tag{8}$$

of the boost velocity vector of the transformation should be fixed, independently of t. However the connection will not be preserved if there is a non-vanishing value for the corresponding relative acceleration vector a^μ as given by

$$a^i = \frac{db^i}{dt}\,, \qquad a^0 = 0\,. \tag{9}$$

It can be verified that for the new section obtained by a generic (i.e. accelerated) Milne gauge transformation the corresponding new connection will be related to the original one by a relation of the simple but non-trivial form $\Gamma_{\mu}{}^{\nu}{}_{\rho} \mapsto \breve{\Gamma}_{\mu}{}^{\nu}{}_{\rho}$ with

$$\breve{\Gamma}_{\mu}{}^{\nu}{}_{\rho} = \Gamma_{\mu}{}^{\nu}{}_{\rho} - t_\mu a^\nu t_\rho\,, \tag{10}$$

where a^ν is the relevant transformation generator, which can be any vector that is spacelike and spacially uniform (i.e. purely time dependent) in the sense that

$$t_\mu a^\mu = 0\,, \qquad \gamma^{\mu\nu}\nabla_\nu a^\rho = 0\,. \tag{11}$$

This transformation law has the noteworthy feature of preserving the trace of the connection, i.e. it gives

$$\breve{\Gamma}_{\mu}{}^{\nu}{}_{\nu} = \Gamma_{\mu}{}^{\nu}{}_{\nu}\,. \tag{12}$$

It follows that if n^μ is a physically well defined current 4-vector of the kind to be discussed in the next section then the divergence given by the expression $\nabla_\nu n^\nu$ will also be physically well defined as a gauge independent scalar field (which will vanish in the particular case of a current that is conserved).

It will be convenient for future reference to introduce a scalar *boost potential* function β that is defined, modulo an arbitrary time dependent constant of integration $\beta\{0\}$ by,

$$b^\mu = \gamma^{\mu\nu}\nabla_\nu\beta\,, \qquad \gamma^{\mu\nu}\nabla_\nu b^\rho = 0\,, \tag{13}$$

so that in the original Aristotelian coordinate system it will be given explicitly by an expression of the form

$$\beta = \beta\{0\} + \gamma_{ij} b^i X^j \,, \qquad \gamma^{\mu\nu}\nabla_\nu \beta\{0\} = 0 \,. \tag{14}$$

It then follows that the relative acceleration vector will be given by

$$a^\mu = e^\nu \nabla_\nu b^\mu = \gamma^{\mu\nu}\nabla_\nu \,\alpha \,, \qquad \alpha = e^\nu \nabla_\nu \beta \,, \tag{15}$$

where e^μ is the relevant "ether" 4-velocity vector, i.e. the unit time lapse vector of the Aristotelian rest frame, whose components with respect to the corresponding coordinates $\{t, X^i\}$ will be given by $e^0 = 1$, $e^i = 0$, so that it will satisfy the conditions

$$e^\mu t_\mu = 1 \,, \qquad \nabla_\mu e^\nu = 0 \,. \tag{16}$$

Whichever bundle section may have been used to specify it in the first place, the connection will automatically be such as to preserve the fundamental tensors $\gamma^{\mu\nu}$ and t_μ of the (Coriolis) spacetime structure, i.e. it will satisfy

$$\nabla_\mu \gamma^{\nu\rho} = 0 \,, \qquad \nabla_\mu t_\nu = 0 \,, \tag{17}$$

and hence also

$$\nabla_\mu \varepsilon^{\nu\rho\sigma\tau} = 0 \,. \tag{18}$$

Conversely the specification of any particular connection that satisfies these preservation conditions will characterize what may be described as the corresponding Galilei structure, which can be conceived as an equivalence class of Aristotelian (i.e. direct product) structures related by linear gauge transformations of the form (1) with vanishing value of the relative acceleration vector defined by (9).

For centuries after Newton's original development of his theory it was taken for granted that the relevant equations of motion singled out a preferred Galilei structure with a corresponding Galilean transformation group with respect to which their form remained covariant. What Milne realized[14] was that except in the case of a localized system in an asymptotically empty background (such as the solar system example to which the early successes of Newton's theory were restricted) the equivalence principle prevents the prescription of any natural rule for preferring some particular Galilean structure rather than another. Thus, as a consequence of the applicability of the equivalence principle, it transpires that the relevant equations of motion are covariant not just with respect to the Galilei group but also with respect to the larger Milne group, which relates distinct Galilei structures by transformations characterized by non-vanishing values of the relative acceleration vector a^μ.

4. Gravity, Particle Dynamics, and the Newton–Cartan Connection

The way the foregoing considerations apply to the most basic of the Newtonian equations of motion, namely the equation of motion for a free particle in some given gravitational field, is as follows. Having specified the worldline of the particle giving the relevant spacetime coordinates x^μ as functions of the Newtonian time t, one can go on to define the corresponding 4-velocity vector defined as the time parametrized tangent vector given by

$$u^\mu = \frac{dx^\mu}{dt}\,, \tag{19}$$

of which only three components are actually independent since the definition automatically ensures that it satisfies the unit normalization condition

$$u^\mu t_\mu = 1\,. \tag{20}$$

The equation of motion will then be expressible covariantly as

$$u^\nu \nabla_\nu u^\mu = g^\mu\,, \tag{21}$$

where g^μ is the relevant gravitational field vector, which must be strictly spacelike to avoid inconsistency with (20), and which, more specifically, is required to be derivable as the space gradient of a Newtonian potential ϕ, i.e.

$$t_\mu g^\mu = 0\,, \qquad g^\mu = -\gamma^{\mu\nu}\nabla_\nu \phi\,. \tag{22}$$

Although it was traditionally expressed in a non-covariant mathematical form, the relation (21) was recognised from the outset to be physically invariant with respect to Galilean transformations, i.e. the linear transformations that preserve the connection involved in the covariant differentiation operator ∇_ν. The crucial point that eluded Newton and everyone else before the time of Einstein, Friedmann, and Milne is that g^μ is akin to the scalar potential ϕ (and to Maxwell's covector potential A_μ as contrasted with the tensorially well defined electromagnetic field $F_{\mu\nu}$) in that it cannot be considered to be an absolutely well defined locally measurable vector field but should be recognised to be gauge dependent. It can be seen from Eq. (10) that the relation (21) is in fact invariant not just with respect to Galilean transformations but to generic Milne transformations as characterized by a non-vanishing acceleration vector a^μ in (9), provided it is understood that g^μ undergoes a corresponding Milne gauge transformation of the simple form $g^\mu \mapsto \breve{g}^\mu$ with

$$\breve{g}^\mu = g^\mu - a^\mu\,. \tag{23}$$

This evidently entails the requirement that the Newtonian potential should transform according to a law of the form $\phi \mapsto \breve{\phi}$ with

$$\breve{\phi} = \phi + \alpha\,, \tag{24}$$

in which scalar field α will be given by (15) as the ether (i.e. Aristotelian) frame time derivative of the boost potential β. The freedom to adjust the specification (14) of the latter by freely choosing the time dependence of the value $\beta\{0\}$ of the boost potential at the Aristotelian coordinate origin corresponds to the calibration freedom in the specification of ϕ by (22). Thus, even in a fixed Aristotelian frame, the potential ϕ will be subject to trivial gauge transformations consisting of the addition of a purely time dependent quantity that is identifiable simply as the time derivative of $\beta\{0\}$.

The idea of the Newton–Cartan formulation[11,12] is to replace the gauge dependent differential operator ∇_μ by a corresponding gauge covariant differential operator D_μ that is specified by replacing the flat connection $\Gamma_{\mu\ \rho}^{\ \nu}$ by a gravitationally modified connection $\omega_{\mu\ \rho}^{\ \nu}$ say that is given by

$$\omega_{\mu\ \rho}^{\ \nu} = \Gamma_{\mu\ \rho}^{\ \nu} - t_\mu g^\nu t_\rho \,. \tag{25}$$

Using (23) in conjunction with (10), it can be verified that this modified connection has the desired gauge invariance property $\omega_{\mu\ \rho}^{\ \nu} \mapsto \breve{\omega}_{\mu\ \rho}^{\ \nu}$ with

$$\breve{\omega}_{\mu\ \rho}^{\ \nu} = \omega_{\mu\ \rho}^{\ \nu} \,. \tag{26}$$

This makes it possible to rewrite the dynamical equation (21) in the manifestly gauge invariant (as well as coordinate covariant) form

$$u^\nu D_\nu u^\mu = 0 \,. \tag{27}$$

While facilitating the comparison with general relativity, this gauge covariant differention operator D_μ has the disadvantage of lacking the convenient flatness property (7) of the gauge dependent alternative ∇_μ. For example if we consider not just a single particle trajectory but a fluid flow characterized by a velocity 4-vector u^μ that is defined as a field over spacetime then we shall have

$$D_{[\mu} D_{\nu]} u^\rho = \frac{1}{2} R_{\mu\nu\ \sigma}^{\ \ \rho} u^\sigma \,, \tag{28}$$

using square brackets to indicate index antisymmetrization, where $R_{\mu\nu\ \sigma}^{\ \ \rho}$ is the Newton–Cartan curvature tensor which can easily be seen to be given by the expression

$$R_{\mu\nu\ \sigma}^{\ \ \rho} = 2 t_\sigma t_{[\mu} \nabla_{\nu]} g^\rho \,. \tag{29}$$

Although it is not immediately manifest from this formula, it can be easily verified using Eqs. (11) and (23) that this curvature tensor is indeed invariant under the gauge transformation (10), i.e. it satisfies $R_{\mu\nu\ \sigma}^{\ \ \rho} \mapsto \breve{R}_{\mu\nu\ \sigma}^{\ \ \rho}$ with

$$\breve{R}_{\mu\nu\ \sigma}^{\ \ \rho} = R_{\mu\nu\ \sigma}^{\ \ \rho} \,. \tag{30}$$

It can be seen that the corresponding (again Milne gauge independent) Ricci type curvature trace tensor is proportional to the Laplacian of the gravitational potential, having only a single independent component that is specified by the formula

$$R_{\mu\nu} = R_{\rho\mu\ \nu}^{\ \ \rho} = t_\mu t_\nu \gamma^{\rho\sigma} \nabla_\rho \nabla_\sigma \phi \,. \tag{31}$$

5. Action and the 4-Momentum Covector

Although, as has just been shown, the dynamical equation (27) is gauge independent, its derivation from an action principle requires the use of a gauge dependent momentum covector π_μ that is in many ways analogous to the electromagnetic gauge potential A_μ that is needed for the variational formulation of Maxwell's equation. The original variational formulation of the Newtonian dynamical equation by Laplace was given by an action integral

$$\mathcal{I} = \int L\, dt\,, \tag{32}$$

in which the Lagrangian scalar is taken to be the difference between the kinetic and potential energies in the well-known form

$$L = \frac{1}{2} m v^2 - m\phi\,, \tag{33}$$

where m is a constant mass parameter, and v is the magnitude of the 3 velocity vector v^μ as specified with respect to some chosen Aristotelian reference system, in terms of the corresponding ether vector e^μ, by

$$v^\mu = u^\mu - e^\mu\,, \qquad v^\mu t_\mu = 0\,, \tag{34}$$

so that its Aristotelian components will be given by

$$v^0 = 0\,, \qquad v^i = \frac{dX^i}{dt}\,. \tag{35}$$

and its magnitude by

$$v^2 = \gamma_{ij} v^i v^j\,. \tag{36}$$

This action can be rewritten in the more elegantly covariant form

$$\mathcal{I} = \int \pi_\mu\, dx^\mu\,, \tag{37}$$

which is evidently equivalent to taking

$$L = \pi_\mu u^\mu\,, \tag{38}$$

by defining the appropriate gauge dependent 4-momentum covector as follows. In terms of the Aristotelian coordinate system $x^0 = t$, $x^i = X^i$ characterizing the chosen gauge, in which we shall evidently have

$$u^0 = 1\,, \qquad u^i = v^i\,, \tag{39}$$

the appropriate 4-momentum covector will be given by the prescription

$$\pi_0 = -\mathcal{E}\,, \qquad \pi_i = m\gamma_{ij} v^j\,, \tag{40}$$

in which the total particle energy is given by the usual formula

$$\mathcal{E} = \frac{1}{2} m v^2 + m\phi\,. \tag{41}$$

Instead of explicit reliance on the Aristotelian coordinate system, we can use the corresponding ether vector e^μ as introduced in Eq. (16), i.e. the unit 4-velocity vector of the Aristotelian rest frame, with respect to which its components will be given simply by

$$e^0 = 1\,, \qquad e^i = 0\,, \tag{42}$$

for the purpose of obtaining a covariant expression for π_μ as follows. To start with, we use the Kronecker unit tensor $\delta^\mu_{\ \nu}$ to construct the corresponding Aristotelian space projection tensor according to the specification

$$\gamma_\nu^{\ \mu} = \delta^\mu_{\ \nu} - e^{\ \mu} t_\nu\,. \tag{43}$$

We then use the defining relations

$$\gamma_{\mu\nu} e^\nu = 0\,, \qquad \gamma_{\mu\nu}\gamma^{\nu\rho} = \gamma^\rho_\mu\,, \tag{44}$$

to characterize the covariant tensor $\gamma_{\mu\nu}$ obtained via the Aristotelian coordinate mapping from the space metric γ_{ij} in the constant time 3-sections. We can then define the kinetic 4-momentum vector by

$$p_\mu = m v_\mu - \frac{1}{2} m v^2 t_\mu\,, \tag{45}$$

where

$$v_\mu = \gamma_{\mu\nu} u^\nu\,, \qquad v^2 = v_\mu v^\mu = \gamma_{\mu\nu} u^\mu u^\nu \tag{46}$$

so that we have

$$\frac{1}{2} m v^2 = p_\nu u^\nu = -p_\nu e^\nu\,. \tag{47}$$

Like the u^μ the kinetic momentum has only three independent components, being subject to a constraint that in this case (unlike that of u^μ) is ether frame dependent, having the form

$$\gamma^{\mu\nu} p_\mu p_\nu = -2m e^\mu p_\mu\,. \tag{48}$$

In terms of this kinetic contribution we finally obtain the expression

$$\pi_\mu = p_\mu - m\phi\, t_\mu\,, \tag{49}$$

for the complete momentum covector.

6. Finite and Infinitesimal Gauge Transformation Rules

Since the boost transformation law for the Aristotelian ether vector evidently takes the form $e^\mu \mapsto \breve{e}^\mu$ with

$$\breve{e}^\mu = e^\mu + b^\mu\,, \tag{50}$$

it can be seen that while, the degenerate contravariant metric tensor is gauge invariant,

$$\breve{\gamma}^{\mu\nu} = \gamma^{\mu\nu}\,, \tag{51}$$

the corresponding mixed projection tensor (43) will undergo a transformation given by

$$\breve{\gamma}^{\mu}{}_{\nu} = \gamma^{\mu}{}_{\nu} - t_\nu b^\mu \,, \tag{52}$$

and the corresponding degenerate covariant metric tensor will be governed by the less trivial transformation rule

$$\breve{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - 2t_{(\mu}\gamma_{\nu)\rho} b^\rho + b^2 t_\mu t_\nu \,, \tag{53}$$

in which the boost magnitude b is naturally defined by

$$b^2 = \gamma_{\mu\nu} b^\mu b^\nu \,. \tag{54}$$

Thus while the contravariant form of the relative velocity (34) obeys the simple Galilean transformation rule

$$\breve{v}^\mu = v^\mu - b^\mu \,, \tag{55}$$

the corresponding covariant vector

$$v_\mu = \gamma_{\mu\nu} v^\nu \tag{56}$$

transforms according to the less simple rule

$$\breve{v}_\mu = v_\mu - \gamma_{\mu\nu} b^\nu + t_\mu (b^2 - b^\nu v_\nu) \,, \tag{57}$$

while for the squared velocity

$$v^2 = v^\mu v_\mu = \gamma_{\mu\nu} u^\mu u^\nu \,, \tag{58}$$

one obtains

$$\breve{v}^2 = v^2 - 2b_\mu v^\mu + b^2 \,. \tag{59}$$

When applied to the kinetic momentum covector (45) the foregoing formulae provide the gauge transformation rule $p_\mu \mapsto \breve{p}_\mu$ with

$$\breve{p}_\mu = p_\mu - m\gamma_{\mu\nu} b^\nu + \frac{1}{2} m b^2 t_\mu \,, \tag{60}$$

and thus the transformation rule for the complete momentum covector (49) can be seen to be expressible in terms of the boost potential β by

$$\breve{\pi}_\mu = \pi_\mu - m\nabla_\mu \beta + \frac{1}{2} m b^2 t_\mu \,, \tag{61}$$

so that, in particular, the corresponding transformation law for its energy component (41) will be expressible as

$$\breve{\mathcal{E}} = \mathcal{E} - p_\nu b^\nu + m\left(\frac{1}{2} b^2 + \alpha\right) , \tag{62}$$

while finally for the Lagrangian scalar (38) we obtain $L \mapsto \breve{L}$ with

$$\breve{L} = L - m u^\mu \nabla_\mu \beta + \frac{1}{2} m b^2 \,. \tag{63}$$

It is apparent from the form of these transformation rules that it will be convenient to work with a recalibrated boost potential, $\hat{\beta} = \beta - \frac{1}{2}\int b^2 dt$, that will be characterized by

$$\nabla_\nu \hat{\beta} = \nabla_\nu \beta - \frac{1}{2} b^2 t_\nu \,. \tag{64}$$

Since the difference between β and $\hat{\beta}$ is a function only of the cosmological time, we can just as well use the latter as the former in the characterization (13), which can be rewritten as

$$b^\mu = \gamma^{\mu\nu} \nabla_\nu \hat{\beta} \,, \qquad \gamma^{\mu\nu} \nabla_\nu b^\rho = 0 \,, \tag{65}$$

but in terms of the recalibrated boost potential the expression (15) for the acceleration will acquire the slightly modified form

$$a^\mu = \gamma^{\mu\nu} \nabla_\nu \alpha \,, \qquad \alpha = e^\mu \nabla_\mu \hat{\beta} + \frac{1}{2} b^2 \,. \tag{66}$$

The corresponding modification of the formula (62) for the transformation of the energy will be given by

$$\breve{\mathcal{E}} = \mathcal{E} - p_\nu b^\nu + m(e^\nu + b^\nu) \nabla_\nu \hat{\beta} \,. \tag{67}$$

However it is for the 4-momentum covector and the Lagrangian that the advantage of the modified boost potential becomes apparent, since it allows Eqs. (61) and (62) to be rewritten more simply as

$$\breve{\pi}_\mu = \pi_\mu - m \nabla_\mu \hat{\beta} \,, \tag{68}$$

and

$$\breve{L} = L - m u^\mu \nabla_\mu \hat{\beta} \,. \tag{69}$$

The effect of the gauge transformation on the action integral (32) can thus be seen to be given by $\mathcal{I} \mapsto \breve{\mathcal{I}}$ with

$$\breve{\mathcal{I}} = \mathcal{I} - m[\hat{\beta}] \,, \tag{70}$$

using large square brackets to denote the total change in the boost potential β along the worldline segment under consideration. Since the gauge adjustment term in Eq. (70) will not be affected by any purely local variation of the worldine (local meaning that it is non-vanishing only on a confined subsegment lying entirely inside the extended worldline segment under consideration) it is obvious that it will have no effect as far as the application of the variational principle is concerned. The observation that the gauge transformation changes the action only by an amount that is constant in the sense of being independent of local worldline variations evidently accounts for the invariance with respect to these (linear Galilean or accelerated Milne) transformations of the ensuing system of dynamical equations.

The preceeding example is one of many in which it is simpler not to work with finite (Galilean or Milne) transformations as we have been doing so far, but with linearized infinitesimal transformations. For a given gauge transformation generated by a given boost potential β, whereby a generic quantity, q say, is subject to a mapping $q \mapsto \breve{q}$, the corresponding infinitesimal gauge transformation $q \mapsto \breve{\mathrm{d}}q$ is defined by a routine two step procedure as follows. The first step is to construct a homotopic interpolation by a one parameter family of gauge transformations $q \mapsto q\{\epsilon\}$ with $q\{0\} = q$ and $q\{1\} = \breve{q}$, for which $q\{\epsilon\}$ is given, for intermediate values of the homotopy parameter ϵ, by an interpolating boost potential $\beta\{\epsilon\} = \epsilon\beta$. The corresponding infinitesimal transformation is then obtained by taking the limit, as $\epsilon \to 0$, of the derivative with respect to ϵ, so that we have

$$\breve{\mathrm{d}}q = \lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(q\{\epsilon\})\,. \tag{71}$$

This differential operation will merely restore the value we started with (as recovered by setting $\epsilon = 0$) for quantities whose dependence on the transformation amplitude is homogeneously linear as is the case for the diverse derivatives of the boost function, which will be characterized by

$$\breve{\mathrm{d}}\alpha = \alpha\,, \qquad \breve{\mathrm{d}}b^\mu = b^\mu\,, \qquad \breve{\mathrm{d}}a^\mu = a^\mu\,, \tag{72}$$

and also of course for the original boost function β itself, though not for the modified boost function $\hat\beta$ for which we obtain a convenient simplification,

$$\breve{\mathrm{d}}\hat\beta = \breve{\mathrm{d}}\beta = \beta\,, \tag{73}$$

which means that at the differential level the modification $\hat\beta$ is redundant. In a similar manner, going to the differential level provides no great simplification for quantities whose gauge transformation depends just linearly on the boost amplitude, as is the case for ether vector e^μ, and the 3-velocity vector v^μ, for which we obtain

$$\breve{\mathrm{d}}e^\mu = b^\mu\,, \qquad \breve{\mathrm{d}}v^\mu = -b^\mu\,, \tag{74}$$

as well as for the less trivial cases of the mixed projection tensor γ^μ_ν, and the connection $\Gamma_\mu{}^\nu{}_\rho$, for which, from (52) and (10), we obtain

$$\breve{\mathrm{d}}\gamma^\mu_\nu = -t_\nu b^\mu\,, \qquad \breve{\mathrm{d}}\Gamma_\mu{}^\nu{}_\rho = -t_\mu a^\nu t_\rho\,, \tag{75}$$

but for quantities with more complicated gauge dependence the differential level is much more convenient. Noteworthy examples are the covariant space metric tensor $\gamma_{\mu\nu}$ and the kinetic momentum vector p_μ for which the formulae (53) and (60) reduce simply to

$$\breve{\mathrm{d}}\gamma_{\mu\nu} = -2t_{(\mu}\gamma_{\nu)\rho}b^\rho\,, \qquad \breve{\mathrm{d}}p_\mu = -m\gamma_{\mu\nu}b^\nu\,, \tag{76}$$

and particularly the Lagrangian L, and the complete momentum covector π_μ, for which we simply obtain

$$\breve{\mathrm{d}}L = -mu^\nu\nabla_\nu\beta\,, \qquad \breve{\mathrm{d}}\pi_\mu = -m\nabla_\mu\beta\,. \tag{77}$$

(The feature of transforming just by the addition of the gradient of a scalar is a property that the 4-momentum covector π_μ of Newtonian dynamics shares with the covector potential A_μ of Maxwellian electromagnetism, a relationship that is essential for the amalgamation of gravity and electromagnetism in the corresponding general relativistic theory.)[3]

The corresponding differential version of the formula (70) for the action integral is given by the expression

$$\breve{\mathrm{d}}\mathcal{I} = -m[\beta]\,, \tag{78}$$

whose evident path independence makes it obvious that the ensuing dynamical theory will have to be gauge invariant, as shown above by the existence of the manifestly covariant formulation (27) of the equations of motion.

7. Covariant Fluid Current Variation Formulae

The simple Lagrangian worldline variation principle for a single particle that was discussed in the preceeding sections can be generalized in an obviously natural way to a flow line variation principle for a fluid system.

Before proceeding, it is to be recalled that in the absence of any prescribed spacetime structure the usual description of currents in terms of vector fields will not be available, but it will still be possible to use the more fundamental Cartan type description whereby a current is represented as a 3-form, i.e. an antisymmetric covariant tensor with four-dimensional components $N_{\mu\nu\rho}$ whose surface integral

$$N = \frac{1}{3!}\int N_{\mu\nu\rho}\, d^3x^{\mu\nu\rho} \tag{79}$$

determines the total number flux over a 3-surface with tangent element

$$d^3x^{\mu\nu\rho} = 3!d_{(1)}x^{[\mu}d_{(2)}x^{\nu}d_{(3)}x^{\rho]}$$

generated by infinitesimal displacements $d_{(i)}x^\mu$ $(i = 1, 2, 3)$. In such a description, the condition for the conservation of the number flux is the vanishing of its exterior derivative as defined by $(\partial \wedge N)_{\mu\nu\rho\sigma} = 4\partial_{[\mu}N_{\nu\rho\sigma]}$. A convenient feature of such exterior differentiation is that it makes no difference if the partial differentiation operator ∂_μ is replaced by a tensorially covariant differentiation operator ∇_μ (or by Cartan's gauge covariant differentiation operator D_μ) since due to the antisymmetrization all the (symmetric) connection components will cancel out.

The more fundamental three index description of the current will of course be replacable by a more compact description involving a single contravariant index whenever the spacetime background is endowed with a canonical antisymmetric measure tensor $\varepsilon_{\mu\nu\rho\sigma}$ and a corresponding contravariant alternating tensor $\varepsilon^{\mu\nu\rho\sigma}$, which, as seen above, will be the case both in relativistic and Newtonian spacetime. The required current vector, with Aristotelian rest frame components

$$n^0 = n\,, \qquad n^i = nv^i\,, \tag{80}$$

where n is the ordinary (scalar) particle number density, will then be given by the duality relation

$$n^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}N_{\nu\rho\sigma}\,, \tag{81}$$

which can be inverted to provide the expression

$$N_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}n^\sigma\,. \tag{82}$$

For *any* covariant symmetric connection compatible with the measure preservation condition (18), the corresponding covariant differentiation operator will determine a divergence that will just be the dual of the exterior derivative operator. Thus independently of the choice of gauge one obtains the equivalent expressions

$$\nabla_\mu n^\mu = D_\mu n^\mu = \frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}(\partial\wedge N)_{\mu\nu\rho\sigma} = \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}N_{\nu\rho\sigma,\mu}\,, \tag{83}$$

for the particle rate, which will vanish for a current that is conserved.

In order to apply the variational principle, we need to evaluate the variation of the current that will result from transport by the action of an infinitesimal displacement $x^\mu \mapsto x^\mu + \xi^\mu$. As a general principle[3] the fixed point (Eulerian) variation of any field will be given generally by its comoving (Lagrangian) variation minus the relevant Lie derivative. Since the specification of the covariant representation of the current does not depend on any background structure its comoving variation will simply vanish, so its fixed point variation will just be given by the formula

$$\delta N_{\mu\nu\rho} = -\boldsymbol{\xi}\pounds N_{\mu\nu\rho}\,, \tag{84}$$

in which the Lie derivative is given by the standard formula

$$\boldsymbol{\xi}\pounds N_{\mu\nu\rho} = \xi^\sigma\nabla_\sigma N_{\mu\nu\rho} + 3N_{\sigma[\mu\nu}\nabla_{\rho]}\xi^\sigma\,. \tag{85}$$

As for any Lie derivative formula (and as in exterior differentiation) it makes no difference if the partial differentiation operator ∂_μ is replaced by a tensorially covariant differentiation operator ∇_μ (or by Cartan's gauge covariant differentiation D_μ) since due to the antisymmetrization all the (symmetric) connection components will cancel out. Thus by taking the dual of the formula (85) we obtain the useful theorem that for *any* covariant symmetric connection compatible with the measure preservation condition (18), the corresponding covariant differentiation operator ∇_μ can be used to express the fixed point (Eulerian) current variation produced by the displacement vector field ξ^μ in the form

$$\delta n^\mu = n^\nu\nabla_\nu\xi^\mu - \xi^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu\xi^\nu\,. \tag{86}$$

This result establishes the validity in a Newtonian framework of a formula that has long been in regular use in a relativistic context, where it was originally derived by a rather different line of reasoning[3] that depended on the Riemannian specification of the covariant differentiation in terms of the non-degenerate spacetime metric that is no longer available in the Newtonian case. It is clear from this present approach that the formula (86) will remain valid if Cartan's gauge covariant derivative operator D_μ is substituted in place of the flat but gauge dependent derivative operator ∇_μ.

A useful corollary of Eq. (86) is the corresponding formula for the variation of the current divergence, which takes the simple form

$$\delta(\nabla_\nu n^\nu) = -\nabla_\mu(\xi^\mu \nabla_\nu n^\nu)\,. \tag{87}$$

It is immediately apparent from this that if the original current is conserved, i.e. if $\nabla_\nu n^\nu$ vanishes, then the displaced current will have the same conservation property.

The formal identity of the relativistic and Newtonian variation formulae will be lost if we make a decomposition of the usual form

$$n^\mu = n u^\mu\,, \qquad n = n^\mu t_\mu\,, \tag{88}$$

in which n is the ordinary particle number density scalar and u^μ is the 4-velocity of the flow as characterized by the unit normalization condition (20). By contracting (86) with t_μ it can seen that the variation law for the particle number density will be given by

$$\delta n = t_\mu n^\nu \nabla_\nu \xi^\mu - \nabla_\nu(n\xi^\nu)\,, \tag{89}$$

which has a different form from its relativistic analogue[3] due to its dependence on the preferred time basis vector t_μ (instead of the non-degenerate spacetime metric $g_{\mu\nu}$ that plays the corresponding role in the relativistic version). The same remark applies to the corresponding variation of the 4-velocity of the flow, which can be seen from Eqs. (86) and (89) to be given by

$$\delta u^\mu = u^\nu \nabla_\nu \xi^\mu - \xi^\nu \nabla_\nu u^\mu - u^\mu u^\nu t_\rho \nabla_\nu \xi^\rho\,. \tag{90}$$

8. Action Principle for Simple Perfect Fluid

The natural way to extend the single particle action principle discussed above to a corresponding fluid action principle is to base the latter on the spacetime integral of a Lagrangian density Λ that will be given by a decomposition of the form

$$\Lambda = \Lambda_{\rm pot} + \Lambda_{\rm kin} + \Lambda_{\rm int}\,, \tag{91}$$

in which the first two (gauge dependent) terms are given by the product of the relevant particle number density n with the corresponding single particle contributions, while the extra term $\Lambda_{\rm int}$ is given simply by

$$\Lambda_{\rm int} = -U_{\rm int}\,, \tag{92}$$

where $U_{\rm int}$ is the internal compression energy, which will be a (naturally gauge independent) function just of the particle number density n. Specifically, the external potential energy contribution will be given by usual Newtonian formula

$$\Lambda_{\rm pot} = -nm\phi\,, \tag{93}$$

which may be rewritten in covariant form as

$$\Lambda_{\rm pot} = -\phi\, \rho^\mu t_\mu \,, \tag{94}$$

where the mass density current is defined by

$$\rho^\mu = \rho u^\mu = m n^\mu \,, \qquad \rho = nm \,, \tag{95}$$

while in accordance with (47) the kinetic contribution will be given by

$$\Lambda_{\rm kin} = n p_\mu u^\mu = n^\mu p_\mu \,, \tag{96}$$

in terms of the (gauge dependent) kinetic 4-momentum covector defined by (45). The extra (gauge independent) contribution (92) representing the negative of the internal compression energy density will determine a corresponding (gauge independent) chemical potential χ by a variation formula of the standard form

$$\delta U_{\rm int} = \chi \delta n \,. \tag{97}$$

In terms of this chemical potential function, the relevant perfect fluid pressure function P (which is also gauge independent) will be given by the well known formula

$$P = n\chi - U_{\rm int} \,. \tag{98}$$

Since, as a consequence of the restriction (48), the variation of the kinetic momentum is automatically constrained to satisfy the identity

$$u^\mu \delta p_\mu = 0 \,, \tag{99}$$

it follows that the generic variation of the purely kinetic contribution to the Lagrangian will be given simply by

$$\delta\Lambda_{\rm kin} = p_\mu \delta n^\mu \,, \tag{100}$$

The variation of the combination (91) will therefore be given by an expression of the canonical form

$$\delta\Lambda = \pi_\mu \delta n^\mu - \rho \delta\phi \,, \tag{101}$$

in which the total 4-momentum is given by an expression of the form

$$\pi_\mu = \mu_\mu - m\phi\, t_\mu \,, \tag{102}$$

which differs from the corresponding free particle momentum formula (49) by the replacement of the purely kinetic contribution p_μ by a total material 4-momentum covector μ_μ that is defined by

$$\mu_\mu = p_\mu - \chi t_\mu \,. \tag{103}$$

This material momentum covector is alternatively definable by the variation formula

$$\delta\Lambda_{\rm mat} = \mu_\mu \delta n^\mu \,, \tag{104}$$

where the total material (non-gravitational) contribution to the Lagrangian density is defined by

$$\Lambda_{\rm mat} = \Lambda_{\rm kin} + \Lambda_{\rm int} \,. \tag{105}$$

It can now be seen that the complete Lagrangian (91) will be elegantly expressible in terms of the 4-momentum covector π_μ and the pressure function P as

$$\Lambda = n^\mu \pi_\mu + P \,. \tag{106}$$

It is to be remarked that while the pressure term in (106) is gauge invariant, the first term is not. However as the formula (102) for the fluid particle 4-momentum covector π_μ differs from its single particle analogue (49) only by the gauge independent term proportional to χ in (103), it can be seen its variation $\breve{\rm d}\pi_\mu$ under the action of an infinitesimal gauge transformation will be given by the same simple formula (77) as in the single particle case. It follows that the corresponding infinitesimal gauge variation of the Lagrangian density (106) will be given simply by

$$\breve{\rm d}\Lambda = -\rho^\nu \nabla_\nu \beta \,. \tag{107}$$

If it is taken for granted that the fluid obeys the ordinary Newtonian mass conservation law

$$\nabla_\nu \rho^\nu = 0 \,, \tag{108}$$

the infinitesimal gauge variation will be expressible as a pure divergence in the form

$$\breve{\rm d}\Lambda = -\nabla_\nu \big(\beta \rho^\nu\big) \,. \tag{109}$$

This means that the gauge change will have no effect on a localized variation of the spacetime volume integral of the action density, so the dynamical equations given by the action principle will automatically be gauge independent.

For the actual evaluation of the variation of the action, the work of the preceeding section provides all the elements that are needed. It can be seen from Eqs. (101) and (86) that when the fluid flow is subjected to the action of an infinitesimal displacement vector field ξ^μ, the resulting variation of the Lagrangian density will be given by the formula

$$\delta\Lambda = \nabla_\mu \big(2\pi_\nu n^{[\mu} \xi^{\nu]}\big) - f_\mu \xi^\mu - \rho \delta\phi \,, \tag{110}$$

in which the covector f_μ is interpretable as the 4-force density acting on the fluid, excluding the gravitational contribution which is already taken into account within the formalism. This force density can be seen to be given by the prescription

$$f_\mu = 2 n^\nu \nabla_{[\nu} \pi_{\mu]} + \pi_\mu \nabla_\nu n^\nu \,, \tag{111}$$

i.e. it is constructed from the current vector and the corresponding momentum covector by contraction with (exterior) derivative plus derivative of contraction.

The postulate that the mass parameter m should be constant means that the mass conservation law (108) will be equivalent to the particle conservation law given, in accordance with Eq. (83) by

$$\nabla_\nu n^\nu = 0\,, \tag{112}$$

a result that is alternatively derivable, wherever the product $n^\mu \pi_\mu$ is non-zero, as a consequence of the variational requirement that the force density (111) should vanish. Subject to Eq. (112), the expression for the force density will evidently reduce to the simple form

$$f_\mu = n^\nu \varpi_{\nu\mu}\,, \tag{113}$$

in which the relevant generalized 4-vorticity 2-form is defined, as the exterior derivative of the 4-momentum covector, by

$$\varpi_{\mu\nu} = 2\nabla_{[\mu}\pi_{\nu]}\,. \tag{114}$$

This vorticity 2-form $\varpi_{\mu\nu}$ is generalized in the sense that it automatically includes allowance for gravity, whose effect can be separated out in the decomposition

$$\varpi_{\mu\nu} = w_{\mu\nu} + 2mt_{[\mu}\nabla_{\nu]}\phi\,. \tag{115}$$

in which $w_{\mu\nu}$ is the ordinary material vorticity tensor defined by

$$w_{\mu\nu} = 2\nabla_{[\mu}\mu_{\nu]}\,. \tag{116}$$

The adjustment allowing for gravitation affects only the time components, so both the complete and the material vorticity give the same purely spacelike 3-velocity vector, which is expressible independently of ϕ as

$$w^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho} w_{\nu\rho} = \frac{1}{2}\varepsilon^{\mu\nu\rho}\varpi_{\nu\rho}\,, \tag{117}$$

and which is related to the purely kinematic local angular velocity vector ω^μ by the proportionality relation

$$w^\mu = 2m\omega^\mu\,, \qquad \omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho}\nabla_\nu v_\rho\,. \tag{118}$$

A related quantity that is easy to analyse in the covariant formalism we are using here, but much more awkward to treat using the traditional $3+1$ spacetime decomposition — so much so that its role in Newtonian fluid dynamics, was not recognised until the relatively recent work of Moreau[15] and Moffat[16] (a century after the pionnering analysis of vorticity by workers of Kelvin's generation) is that of helicity. In that work (and in its more recent non-barotropic generalization)[17] the helicity was introduced as a scalar density that was constructed as the three dimension scalar product of the velocity vector v^i and the vorticity vector w^i. On the basis of experience[3] with the relativistic case, it is evident that in the 4-dimensionally covariant formalism we are using here, the helicity will most naturally be definable[2,1] as a vectorial current η^μ that is proportional to the dual of the exterior product of the energy momentum covector π_μ with the corresponding generalized vorticity two form, $\varpi_{\mu\nu}$, namely

$$\eta^\mu = \varepsilon^{\mu\nu\rho\sigma}\pi_\nu\nabla_\rho\pi_\sigma = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\pi_\nu\varpi_{\rho\sigma}\,. \tag{119}$$

The time component of this quantity can be seen from Eq. (6) to be proportional to the Moreau–Moffat helicity scalar, with a negative coefficient (in the sign convention we are using, for which the sign of the measure component ε_{0123} is taken to be positive, so that of ε^{0123} will be negative) that is given by $-2m^2$, i.e.

$$\eta^\mu t_\mu = -w^\mu\pi_\mu = -2m^2\omega^i v_i\,. \tag{120}$$

It immediately follows from the Eulerian dynamical equation, whose variational derivation will be described below, that this helicity current η^μ will be conserved in the simple sense that its 4-divergence,

$$\nabla_\mu\eta^\mu = \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}\varpi_{\mu\nu}\varpi_{\rho\sigma}\,, \tag{121}$$

will turn out to vanish, a property that is both obvious and easy to express in the four-dimensional approach used here, but not so trivial, either to derive or even to present, within the (Latin as opposed to Greek index) framework of the traditional $3+1$ formalism (see appendix).

The purport of the variational principle is that the spacetime volume integral of $\delta\Lambda$ should vanish for any displacement ξ^μ with bounded support (i.e. that vanishes outside some bounded spacetime region). Since, by Green's theorem, the divergence contribution in (110) will make no contribution to the variational integral, the principle reduces to the requirement that the 4-force density should vanish,

$$f_\mu = 0\,. \tag{122}$$

Subject to Eq. (112) this equation will reduce to the form

$$n^\nu\varpi_{\nu\mu} = 0\,, \tag{123}$$

in which the complete vorticity 2-form can be expressed as

$$\varpi_{\mu\nu} = 2\nabla_{[\mu}p_{\nu]} + 2t_{[\mu}\nabla_{\nu]}(\chi + m\phi)\,. \tag{124}$$

An immediate consequence of Eq. (123) is that (since its components form an antisymmetric matrix with a zero eigenvalue eigenvector, namely n^μ) the vorticity 2-form $\varpi_{\mu\nu}$ must be algebraicly degenerate, with vanishing determinant and therefore matrix rank 2 (since an antisymmetric matrix cannot have even rank), a condition that is expressible by the algebraic restriction

$$\varpi_{\mu[\nu}\varpi_{\rho\sigma]} = 0\,. \tag{125}$$

This has the obvious corollary that the right hand side of Eq. (121) must vanish, and hence that the helicity current η^μ will indeed be conserved, i.e. we shall have

$$\nabla_\mu\eta^\mu = 0\,. \tag{126}$$

Another, more widely known, consequence of the degeneracy property (125) is that (exactly as in the relativistic case explained elsewhere)[3] the vorticity 2-form will be orthogonal to a two-dimensional tangent element, containing the flow vector n^μ as well as the vorticity 3-vector w^μ defined by Eq. (117), and that (as in the analogous case of magnetic 2-surfaces in a perfectly conducting plasma) these two-dimensional tangent elements will be integrable in the sense of meshing together to form well defined vorticity flux 2-surfaces.

The analogue for a 2-form $\varpi_{\mu\nu}$ of the formula (85) for Lie derivation with respect to a flow field n^μ takes the form

$$\mathbf{n}\mathcal{L}\varpi_{\mu\nu} = 3n^\sigma \nabla_{[\sigma}\varpi_{\mu\nu]} - 2\nabla_{[\mu}(\varpi_{\nu]\sigma} n^\sigma)\,, \tag{127}$$

in which the first term will drop out identically by the closure property, i.e. the vanishing of the exterior derivative $3\nabla_{[\sigma}\varpi_{\mu\nu]}$ of the vorticity as an automatic consequence of its exactness property (114). When the field equation (123) is satisfied, it can be seen that the second term in Eq. (127) will also drop out. We thus obtain the covariant generalization of the well known Kelvin vorticity conservation theorem[18] to the effect that the the vorticity 2 form will be conserved by Lie transport, with respect to any arbitrarily rescaled multiple of the flow vector, i.e.

$$(\zeta\mathbf{n})\mathcal{L}\varpi_{\mu\nu} = 0\,, \tag{128}$$

for an arbitrarily variable scalar field ζ.

It is to be noted that if, instead of restricting the variation δn^ν to be given by the worldline displacement formula (86), one merely imposes current conservation by adding a Lagrange multiplier term $\varphi\nabla_\mu n^\mu$ to the action density, then one will get a more restricted dynamical equation to the effect that the momentum covector should be the gradient of the Lagrange multiplier φ and thus that it should be irrotational:

$$\pi_\mu = \nabla_\mu\varphi \quad \Rightarrow \quad \varpi_{\mu\nu} = 0\,. \tag{129}$$

A solution of this irrotational type is the only kind that is allowed in the special case of a simple superfluid (on a mesoscopic — i.e. intervortex — scale) for which the scalar φ is interpretable as being proportional to the quantum phase angle of a bosonic condensate.

In terms of the Newton–Cartan connection given by Eq. (25), the dynamical equation (123) can be rewritten in the manifestly gauge invariant form

$$u^\nu D_\nu u^\mu = -\frac{1}{m}\gamma^{\mu\nu}\nabla_\nu\chi\,. \tag{130}$$

This is just a covariant reformulation of the well-known Euler equation, which is traditionally expressed in terms of the pressure function (98), whose variation can be seen from (97) to be given in terms of that of the chemical potential χ by

$$\delta P = n\delta\chi\,. \tag{131}$$

It is thereby possible to rewrite (130) as

$$u^\nu \nabla_\nu u^\mu = -\gamma^{\mu\nu}\left(\nabla_\nu \phi + \frac{1}{\rho}\nabla_\nu P\right), \tag{132}$$

which can immediately be translated into Aristotelian coordinate notation to give the original Eulerian version in the form[18]

$$\nabla_{_0} v_i + v^j \nabla_j v_i = -\nabla_i \phi - \frac{1}{\rho}\nabla_i P\,. \tag{133}$$

While this last version has the advantage of familiarity, and Eq. (130) has the advantage of manifest gauge covariance (with respect to linear Galilean and non-linear Milne transformations) it is the version (123) that is most convenient for many mathematical purposes, since it involves only exterior differentiation, and can therefore be evaluated in arbitrarily curved (e.g. comoving) coordinates using only partial differentiation, without reference to any of the various relevant connections (the $\omega_\mu{}^\nu{}_\rho$ that is covariant but curved or the connection $\Gamma_\mu{}^\nu{}_\rho$ that is flat but gauge dependent). An example is the demonstration above of the way the use of Eq. (123) greatly facilitates the treatment of helicity conservation, a concept that is almost trivial (actually simpler than the concept of vorticity conservation) in the covariant formalism developed here, but whose original derivation, in the frame dependent notation of Eq. (133) was of a technical complexity such that it was finally obtained (by Moreau and Moffat in the 1960's)[15,16] about a century later than the development of the more elementary precursor concept of vorticity (by nineteenth century pioneers such as Kelvin). The advantage of the fully covariant approach will be even greater when we go on from simple to multiconstituent fluids.

9. Multiconstituent Fluid Models

We now extend the discussion to cases involving several independent — but not always independently conserved — currents with current 4-vectors that we shall denote by $n_{\rm X}{}^\nu$ where X is a "chemical" index with values ranging over the labels of the various constituents involved. In particular the neutron star application for which this work is particularly intended, will involve a neutron number density current $n_{\rm n}{}^\nu$ and a proton number density current $n_{\rm p}{}^\nu$ so in this case the index X will range over the pair of values X = n and X = p. Although the total baryon number current $n_{\rm b} = n_{\rm n} + n_{\rm p}$ will be conserved, in applications dealing with long term evolution the neutron and proton currents will not be separately conserved due to the possibility of transfer of baryons from one to the other by weak interactions. To deal with such cases it may be necessary to allow for the possibility that a particular current $n_{\rm X}{}^\nu$ may be characterized by a non-vanishing value of the destruction rate (per unit spacetime volume) that is defined (as the negative of the corresponding creation rate) by

$$\mathcal{D}_{\rm X} = -\nabla_\nu n_{\rm X}{}^\nu\,, \tag{134}$$

a formula in which, as explained above, it makes no difference what frame may have been used to specify the connection involved in the covariant differentiation operator ∇_ν.

The obviously natural way to set up an appropriate Lagrangian for a multiconstituent fluid model is to take a combination of the same general form

$$\Lambda = \Lambda_{\rm mat} + \Lambda_{\rm pot}\,, \tag{135}$$

as before, with the material contribution $\Lambda_{\rm mat}$ again given by a decomposition of the form (105) as a sum of kinetic and internal contributions. As always, the gravitational potential energy contribution will simply be given by

$$\Lambda_{\rm pot} = -\phi\,\rho\,, \tag{136}$$

where ρ is the total mass density. However this will now be given as a sum over constituents of the form

$$\rho = \sum_{\rm X} m^{\rm X} n_{\rm X}\,, \tag{137}$$

in which $m^{\rm X}$ is the Newtonian mass per particle associated with the current $n_{\rm X}{}^{\nu}$. The prescription (136) evidently has the same general form (94) as before when written covariantly in terms of the corresponding total mass current vector

$$\rho^\nu = \sum_{\rm X} m^{\rm X} n_{\rm X}^{\nu}\,. \tag{138}$$

One of the basic principles of Newtonian theory is that although the different contributions need not be separately conserved (as matter can be transferred from one to another by chemical or nuclear reactions) the total mass current (including all relevant contributions) will still have to obey the conservation law (108).

In summation formulae such as Eqs. (137) and (138) (in which it would be legitimate to use the standard shorthand summation convention whereby the explicit use of the summation symbol Σ is omitted) it is to be noticed that the constituent indices of the masses have been written "upstairs" to indicate their contravariant character with respect to linear constituent recombinations, in contrast with the currents, with indices "downstairs," which undergo recombinations of the corresponding covariant (inverse) form. The formulae (137) and (138) can be seen to be covariant, while the resulting sums ρ and ρ^μ themselves are actually invariant, when such linear transformations of chemical basis are carried out.

A simple illustration of a change of chemical basis is provided by typical astrophysical applications for which it may be sufficient to treat the relevant matter (e.g. in a stellar atmosphere) as a mixture of hydrogen (with atomic nucleus containing just one proton) and helium (with atomic nucleus consisting of 2 protons and 2 neutrons), so that in terms of chemical index values $_{\rm X} = {}_{\rm H}$ and $_{\rm X} = {}_{\rm He}$ the total mass density will be given by

$$\rho = m^{\rm H} n_{\rm H} + m^{\rm He} n_{\rm He}\,.$$

In an equivalent description based on the underlying proton and neutron number densities, using index values $_{\rm X}=$ p and $_{\rm X}=$ n, the total mass density will be given by an expression of the exactly analogous form $\rho = m^{\rm p} n_{\rm p} + m^{\rm n} n_{\rm n}$ in which the relevant densities are provided by a chemical basis transformation that is given by the relations $n_{\rm p} = n_{\rm H} + 2n_{\rm He}$ and $n_{\rm n} = 2n_{\rm He}$. Having voluntarily chosen to use downstairs chemical indices for the currents, we have no option but to use upstairs indices for the masses because their corresponding transformation will be of contravariant kind (i.e. given by the inverse of the covariant transformation matrix) that in this particular illustration will be specified by the relations

$$m^{\rm H} = m^{\rm p}\,, \qquad m^{\rm He} = 2m^{\rm p} + 2m^{\rm n}\,.$$

At the cost perhaps of obscuring other relevant information, such a change of chemical basis (from the atomic reference system to the nuclear reference system) would have the advantage of facilitating the exploitation of the empirical fact that for many practical applications (particularly in contexts for which a Newtonian description is sufficiently accurate) it will be a good enough approximation to take $m^{\rm n} \sim m^{\rm p}$ (a relation attributable to the corresponding, but still theoretically unexplained, approximate inequality between up and down — but not strange — quark masses).

Having dealt with the gravitational potential contribution, we now turn to the kinetic contribution in the decomposition (105) of $\Lambda_{\rm kin}$, for which a multiconstituent version can be obtained simply by adding up the contributions, as specified by (96), of the separate constituents. We thus obtain a prescription of the form

$$\Lambda_{\rm kin} = \sum_{\rm X} n_{\rm X}{}^{\nu} p^{\rm X}{}_{\nu}\,, \tag{139}$$

in which, as in Eq. (138), the sum on the right is covariant with respect to spacetime coordinate transformations. However despite its neat appearance (but due to the non-linearity in the defining formula (45) for relevant kinetic momentum covectors $p^{\rm X}{}_{\nu}$) this contribution (139) is neither gauge invariant (with respect to non-linear Milne or even linear Galilean transformations) nor invariant under changes of chemical basis.

In the manner shown by (70) we can recover gauge invariance of the global integrated action perturbation (though not of the local unperturbed action density) by combining the kinetic contribution with the gravitational potential contribution. This gauge invariance at the global perturbation level will evidently be preserved by the addition of the extra locally Galilei (and hence *a fortiori* also Milne) invariant term $\Lambda_{\rm int}$ that is needed to give a chemically invariant value for the total material contribution $\Lambda_{\rm mat}$, and hence also for the complete Lagrangian (135). The variation

$$\delta\Lambda_{\rm mat} = \sum_{\rm X} \mu^{\rm X}{}_{\nu}\, \delta n_{\rm X}{}^{\nu}\,, \tag{140}$$

of the chemically invariant material Lagrangian will then define a chemically contravariant set of material momentum covectors $\mu^{\rm X}{}_\nu$. These will be decomposible in the form

$$\mu^{\rm X}{}_\nu = p^{\rm X}{}_\nu + \chi^{\rm X}{}_\nu\,, \tag{141}$$

where the internal momentum contributions are defined by the variation

$$\delta\Lambda_{\rm int} = \sum_{\rm X} \chi^{\rm X}{}_\nu\, \delta n_{\rm X}{}^\nu\,, \tag{142}$$

of the internal Lagrangian contribution, whose chemical basis dependence endows the momentum contributions $\chi^{\rm X}{}_\nu$ with corresponding bad (meaning non-contravariant) behavior under chemical basis transformations so as to cancel the bad behavior of the kinetic momentum contributions $p^{\rm X}{}_\nu$ in such a way that the total (141) is chemically well behaved. However, although their chemical transformation behaviour is complicated, the internal momentum contributions $\chi^{\rm X}{}_\nu$ have the convenient redeeming feature that (unlike the chemically well behaved total $\mu^{\rm X}{}_\nu$) they are automatically invariant with respect to Milne (and therefore *a fortiori* Galilean) gauge transformations, as a consequence of the postulated gauge invariance of the Lagrangian contribution $\Lambda_{\rm int}$ itself. In the manner to be derived in Ref. 19, this gauge invariance entails corresponding Noether identities that are expressible as

$$\sum_{\rm X} t_\mu n_{\rm X}{}^\mu \chi^{\rm X}{}_\nu \gamma^{\nu\sigma} = 0\,. \tag{143}$$

and

$$\sum_{\rm X} n_{\rm X}{}^{[\mu}\gamma^{\sigma]\nu}\chi^{\rm X}{}_\nu = 0\,. \tag{144}$$

The space projected parts of the internal momentum give rise to the effect (of a kind that is familiar in the case of ordinary electron currents in a metallic conductor) that is known as "entrainment" whereby the constituent momentum directions may deviate from those of the corresponding velocities. However it follows from (143) that the deviations will cancel out in the total, so that the (gauge dependent but chemically invariant) 3-momentum density, i.e. the space projected part of the 4-momentum density (138), as defined by

$$\Pi^\mu = \gamma^\mu{}_\nu \rho^\nu\,, \tag{145}$$

will be expressible as a sum over separate material momentum contributions in the form

$$\Pi^\mu = \sum_{\rm X} \Pi^{{\rm X}\mu}\,, \tag{146}$$

in which the individual contributions are given by

$$\Pi^{{\rm X}\mu} = n_{\rm X}\mu^{\rm X}{}_\nu \gamma^{\nu\mu}\,. \tag{147}$$

In a manner analogous to that by which the ordinary pressure function P was introduced by Eq. (98) in the simple perfect fluid case, it is useful to define a corresponding gauge invariant (and chemically invariant) generalized pressure function Ψ for the multiconstituent case by the specification

$$\Psi = \Lambda_{\rm int} - \sum_{\rm X} n_{\rm X}{}^{\nu}\chi^{\rm X}{}_{\nu}\,, \tag{148}$$

It can be seen from (142) that the corresponding infinitesimal variation formula will have the form

$$\delta\Psi = -\sum_{\rm X} n_{\rm X}{}^{\nu}\,\delta\chi^{\rm X}{}_{\nu}\,. \tag{149}$$

Putting the kinetic, internal, and external gravitational potential contributions together, we now see that the variation of the complete Lagrangian (135) will have the (chemically covariant) form

$$\delta\Lambda = \sum_{\rm X} \pi^{\rm X}{}_{\nu}\,\delta n_{\rm X}{}^{\nu} - \rho\,\delta\phi\,, \tag{150}$$

while the corresponding variation of the pressure function will be expressible in the form

$$\delta\Psi = -\sum_{\rm X} n_{\rm X}{}^{\nu}\,\delta\pi^{\rm X}{}_{\nu} - \rho\,\delta\phi\,, \tag{151}$$

in which the complete (chemically contravariant) particle momentum covectors of the various constituent currents are given by expressions of the same form (102) as in the single constituent case, namely

$$\pi^{\rm X}{}_{\mu} = \mu^{\rm X}_{\mu} - m^{\rm X}\phi t_{\mu}\,. \tag{152}$$

It is to be observed that since the extra (gravitational) term here is purely temporal, it makes no difference to the three momentum, so Eq. (147) can just as well be written in the form

$$\mathit{\Pi}^{{\rm X}\mu} = n_{\rm X}\pi^{\rm X}{}_{\nu}\gamma^{\nu\mu}\,. \tag{153}$$

As in the case (106) of a simple perfect fluid, we can rewrite the complete Lagrangian in terms of the momenta $\pi^{\rm X}{}_{\nu}$ and the generalized pressure function Ψ in the form

$$\Lambda = \sum_{\rm X} n_{\rm X}{}^{\nu}\pi^{\rm X}{}_{\nu} + \Psi\,. \tag{154}$$

Since, as in the single constituent case, the effect of an infinitesimal gauge transformation on the momenta will be given in terms of the relevant boost potential β by an expression of the simple form

$$\breve{\rm d}\pi^{\rm X}{}_{\mu} = -m^{\rm X}\nabla_{\mu}\beta\,, \tag{155}$$

while the other quantities in (154) will remain invariant, it is evident that the resulting infinitesimal gauge variation of the Lagrangian will be given by the formula

$$\breve{\mathrm{d}}\Lambda = -\sum_{\mathrm{X}} m^{\mathrm{X}} n_{\mathrm{X}}^{\ \nu} \nabla_\nu \beta \, . \tag{156}$$

Subject to the (chemically invariant) restriction that each of the currents should be separately conserved,

$$\nabla_\nu n_{\mathrm{X}}^{\ \nu} = 0 \, , \tag{157}$$

it can be seen to follow that $\breve{\mathrm{d}}\Lambda$ will be expressible as a pure divergence of exactly the same simple form (109) as in the single constituent case. Since it can be seen from Eq. (87) that the restriction (157) is preserved by the generic flow displacement (84) it follows that the corresponding variational equations of motion that we shall derive below will be gauge independent.

It is to be observed that the condition (109) for the gauge invariance of the integrated action will still be satisfied even if the currents do not satisfy the separate conservation conditions (157) but are restricted only by the single conservation condition (108), that must always be satisfied by the total Newtonian mass current (138). However, if the currents are not separately conserved, the total mass conservation condition (108) will not automatically be preserved by a generic set of independent flow displacements of the form (84), so the variational principle will no longer provide an automatically gauge invariant set of field equations.

When each current $n_{\mathrm{X}}^{\ \nu}$ is subject to its own independent displacement $\xi_{\mathrm{X}}^{\ \nu}$, the generalization of Eq. (110) that we finally obtain by substituting Eq. (86) in Eq. (150) will take the form

$$\delta\Lambda = \nabla_\mu \left(2 \sum_{\mathrm{X}} \pi^{\mathrm{X}}{}_{\nu} n_{\mathrm{X}}^{[\mu} \xi_{\mathrm{X}}^{\ \nu]} \right) - \sum_{\mathrm{X}} f^{\mathrm{X}}{}_{\mu} \xi_{\mathrm{X}}^{\ \mu} - \rho \, \delta\phi \, , \tag{158}$$

in which, for each constituent, the covector f^{X}_{μ} is interpretable as the 4-force density acting on the corresponding current $n_{\mathrm{X}}^{\ \mu}$, and in which the value of this force density can be read out as

$$f^{\mathrm{X}}_{\mu} = 2 n_{\mathrm{X}}^{\ \nu} \nabla_{[\nu} \pi^{\mathrm{X}}{}_{\mu]} + \pi^{\mathrm{X}}{}_{\mu} \nabla_\nu n_{\mathrm{X}}^{\ \nu} \, . \tag{159}$$

It is this that must vanish in the strictly conservative case for which the variational field equations are satisfied.

Whenever the separate conservation conditions (157) actually are satisfied, the force density will reduce to the simple form

$$f^{\mathrm{X}}_{\mu} = n_{\mathrm{X}}^{\ \nu} \varpi^{\mathrm{X}}{}_{\nu\mu} \, , \tag{160}$$

using the notation

$$\varpi^{\mathrm{X}}{}_{\mu\nu} = 2 \nabla_{[\mu} \pi^{\mathrm{X}}{}_{\nu]} \, , \tag{161}$$

for the generalized vorticity tensors, which unlike the momenta from which they are derived, can be seen from the form of (155) to be gauge invariant,

$$\check{\mathrm{d}}\varpi^{\mathrm{X}}{}_{\mu\nu} = 0\,. \tag{162}$$

Although the decay rates (134) are also gauge invariant,

$$\check{\mathrm{d}}\mathcal{D}_{\mathrm{X}} = 0\,, \tag{163}$$

whenever they are non-zero, i.e. when the separate conservation conditions (157) are not satisfied, the resulting force density contributions

$$f^{\mathrm{X}}_{\mu} = n_{\mathrm{X}}{}^{\nu}\varpi^{\mathrm{X}}{}_{\nu\mu} - \pi^{\mathrm{X}}{}_{\mu}\mathcal{D}_{\mathrm{X}}\,, \tag{164}$$

will no longer be gauge invariant, but will transform according to the rule

$$\check{\mathrm{d}}f^{\mathrm{X}}_{\mu} = m^{\mathrm{X}}\mathcal{D}_{\mathrm{X}}\nabla_{\mu}\beta\,. \tag{165}$$

It can be seen from the general formula (127) that when the separate current conservation conditions (157) are satisfied, the Lie derivatives of the vorticities with respect to the corresponding flow fields will be given in terms of the corresponding force densities by

$$\mathbf{n}_{\mathrm{X}}\pounds\varpi^{\mathrm{X}}{}_{\mu\nu} = 2\nabla_{[\mu}f^{\mathrm{X}}_{\nu]}\,. \tag{166}$$

When the full set of variational field equations is satisfied, so that the forces densities f^{X}_{μ} all vanish, it can be seen that each vorticity will be conserved by transport along the corresponding flow lines, i.e. each constituent will satisfy a Kelvin type conservation law of the form (128), namely

$$(\zeta^{\mathrm{X}}\mathbf{n}_{\mathrm{X}})\pounds\varpi^{\mathrm{X}}{}_{\mu\nu} = 0\,. \tag{167}$$

for arbitrary scalar fields ζ^{X}.

As in the corresponding relativistic case,[10] one can go on to generalize the single constituent helicity vector (119) to a vector valued helicity matrix defined by

$$\eta^{\mathrm{XY}\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\pi^{\mathrm{X}}{}_{\nu}\varpi^{\mathrm{Y}}{}_{\rho\sigma}\,. \tag{168}$$

The antisymmetric part of this matrix will be exact in the sense of having the form of a divergence,

$$\eta^{[\mathrm{XY}]\mu} = \frac{1}{2}\nabla_{\nu}\left(\varepsilon^{\mu\nu\rho\sigma}\pi^{\mathrm{Y}}{}_{\rho}\pi^{\mathrm{X}}{}_{\sigma}\right), \tag{169}$$

of an antisymmetric tensor, with the implication that it will automatically be closed in the sense that its own divergence will vanish identically, i.e.

$$\nabla_{\mu}\eta^{[\mathrm{XY}]\mu} = 0\,. \tag{170}$$

It follows that if a constituent with label X say is characterized by the property of irrotationality, meaning that $\varpi^{\mathrm{X}}{}_{\mu\nu}$ vanishes, and more particularly, as in Eq. (129), in the case of a constituent that is superfluid, and thus characterized (on a mesoscopic scale) by a momentum covector that is the gradient of a corresponding phase scalar $\varphi^{\rm X}$, so that for any other label value $_{\rm Y}$ say, the corresponding helicity matrix component $\eta^{{\rm YX}\mu}$ will also vanish, i.e. $\eta^{{\rm YX}\mu}=0$, then the corresponding transposed component will automatically be conserved, i.e. we shall have

$$\pi^{\rm X}{}_\mu = \nabla_\mu \varphi^{\rm X} \quad \Rightarrow \quad \varpi^{\rm X}{}_{\mu\nu} = 0 \quad \Rightarrow \nabla_\mu \eta^{{\rm XY}\mu} = 0\,. \tag{171}$$

Regardless of any irrotationality constraint that may be satisfied, it can be seen — for the same reason as in the single constituent case characterized by Eq. (126) — that the vanishing of the force density covector $f^{\rm X}_\mu$ in Eq. (160) will always be sufficient to ensure that the divergence of the corresponding diagonal component of the helicity matrix will vanish, i.e. that the constituent under consideration will be subject to a helicity conservation law[2,10] having the form

$$\nabla_\mu \eta^{{\rm XX}\mu} = 0\,. \tag{172}$$

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Appendix. Helicity Current in the Traditional 3 + 1 Formalism

The efficacity of 4-dimensionally covariant treatment is clearly demonstrated by the ease with which the foregoing helicity conservation laws have been obtained as almost obvious consequences of the dynamical equation. This contrasts with treatment using a 3+1 space versus time decomposition, in which an equivalent derivation of the helicity conservation law[16] requires much greater algebraic effort and ingenuity, as shown in this appendix.

The time and space components of the helicity current are found from Eq. (119) and the specification (6) to be expressible as

$$\eta^0 = -\varepsilon^{ijk}\pi_i \nabla_j \pi_k\,, \tag{A.1}$$

$$\eta^i = \pi_0 \varepsilon^{ijk}\nabla_j \pi_k - \varepsilon^{ijk}\pi_j \nabla_0 \pi_k + \varepsilon^{ijk}\pi_j \nabla_k \pi_0\,. \tag{A.2}$$

In the following — as is usual within an Aristotelian–Cartesian framework — the space indices will be replaced by the familiar arrow notation. We also introduce the cross-product $(\phi\times\varphi)^i$ between two forms ϕ_i and φ_i as the contravariant vector $\varepsilon^{ijk}\phi_j\varphi_k$. The curl and 3-divergence of a vector $\mathbf{V}$ are defined respectively as

$$(\mathbf{curl}\ \mathbf{V})^i = \varepsilon^{ijk}\nabla_j V_k \quad \text{and} \quad \text{div}\ \mathbf{V} = \nabla_i V^i\,.$$

The time derivative will be written as $\nabla_0 = \partial_t$.

With this notation, the helicity 4-vector will be given by

$$\eta^0 = -\boldsymbol{\pi}\cdot\mathbf{curl}\ \boldsymbol{\pi}\,, \tag{A.3}$$

$$\boldsymbol{\eta} = \pi_0\ \mathbf{curl}\ \boldsymbol{\pi} - \boldsymbol{\pi}\times\partial_t\boldsymbol{\pi} + \boldsymbol{\pi}\times\boldsymbol{\nabla}\pi_0\,. \tag{A.4}$$

Considering the example of the single perfect fluid model, the components of the 4-momentum covector (102) are $\pi_0 = -\mathcal{E}$ and $\boldsymbol{\pi} = m\mathbf{v}$, in which we have introduced the total particle energy $\mathcal{E} = \frac{1}{2}mv^2 + m\phi + \chi$. Hence the helicity current reduces to

$$\eta^0 = -2m^2\boldsymbol{\omega}\cdot\mathbf{v}\,, \tag{A.5}$$

$$\boldsymbol{\eta} = -2m\mathcal{E}\boldsymbol{\omega} \tag{A.6}$$

in terms of the kinematic local angular velocity $\boldsymbol{\omega} = \frac{1}{2}\mathbf{curl}\ \mathbf{v}$ (div $\boldsymbol{\omega} = 0$).

The conservation law of this helicity current will now be derived within the $3+1$ spacetime decomposition, starting from Euler's equation in the form[18]

$$m\partial_t\mathbf{v} + \boldsymbol{\nabla}\mathcal{E} + 2m\boldsymbol{\omega}\times\mathbf{v} = 0\,, \tag{A.7}$$

whose contraction with $m\boldsymbol{\omega}$ yields

$$m^2\boldsymbol{\omega}\cdot\partial_t\mathbf{v} + m\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\mathcal{E} = 0\,. \tag{A.8}$$

Taking the curl of the Euler equation followed by the dot-product with the velocity gives

$$m^2\mathbf{v}\cdot\partial_t\boldsymbol{\omega} + m^2\mathbf{v}\cdot\mathbf{curl}(\boldsymbol{\omega}\times\mathbf{v}) = 0\,. \tag{A.9}$$

Adding the two equalities and simplifying, we obtain

$$\begin{aligned}2m^2\partial_t(\boldsymbol{\omega}\cdot\mathbf{v}) + 2\ \mathrm{div}(m\boldsymbol{\omega}\mathcal{E}) + 2m^2(\mathbf{v}\cdot\boldsymbol{\omega})\mathrm{div}\ \mathbf{v}\\ + 2m^2\mathbf{v}\cdot(\mathbf{v}\cdot\boldsymbol{\nabla}\boldsymbol{\omega}) - 2m^2\mathbf{v}\cdot(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\mathbf{v}) = 0\,.\end{aligned} \tag{A.10}$$

Taking the cross-product of the Euler equation then the divergence, using the standard identity

$$\boldsymbol{\nabla}(\mathbf{W}\cdot\mathbf{V}) = \mathbf{W}\times\mathbf{curl}\ \mathbf{V} + \mathbf{V}\times\mathbf{curl}\ \mathbf{W} + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{W} + \mathbf{W}\cdot\boldsymbol{\nabla}\mathbf{V}\,, \tag{A.11}$$

we obtain

$$\begin{aligned}\mathrm{div}(m^2\mathbf{v}\times\partial_t\mathbf{v} + m\mathbf{v}\times\boldsymbol{\nabla}\mathcal{E}) + 8m^2\boldsymbol{\omega}\cdot(\mathbf{v}\times\boldsymbol{\omega}) + 4m^2\boldsymbol{\omega}\cdot(\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v})\\ - 2m^2(\mathbf{v}\cdot\boldsymbol{\omega})\mathrm{div}\ \mathbf{v} - 2m^2\mathbf{v}\cdot(\mathbf{v}\times\mathbf{curl}\ \boldsymbol{\omega})\\ - 2m^2\mathbf{v}\cdot(\mathbf{v}\boldsymbol{\nabla}\boldsymbol{\omega}) - 2m^2\mathbf{v}\cdot(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\mathbf{v}) = 0\,.\end{aligned} \tag{A.12}$$

Combining this with Eq. (A.10), one obtains the spacetime decomposition of Eq. (126), namely the local version of the original Moreau–Moffat helicity conservation law,[15,16] in the form

$$2m^2\partial_t(\boldsymbol{\omega}\cdot\mathbf{v}) = \mathrm{div}(-2m\boldsymbol{\omega}\mathcal{E})\,. \tag{A.13}$$

References

1. B. Carter and I.M. Khalatnikov, *Rev. Math. Phys.* **6**, 277 (1994).
2. Z. Peradzynski, *Int J. Theor. Phys.* **29**, 1277 (1991).
3. B. Carter, "Covariant theory of conductivity in ideal fluid or solid media," in *Relativistic Fluid Dynamics*, C.I.M.E., Noto, May 1987, Lecture Notes in Mathematics 1385, eds. A. M. Anile and Y. Choquet-Bruhat (Springer-Verlag, Heidelberg, 1989), pp. 1–64.
4. D. Langlois, D. Sedrakian and B. Carter, *Mon. Not. R. Astron. Soc.* **297**, 1198 (1998) [astro-ph/9711042].
5. N. Andersson and G. L. Comer, *Class. Quantum Grav.* **18**, 969 (2001) [gr-qc/0009089].
6. B. Carter, D. Langlois and D. Sedrakian, *Astron. & Astrophys.* **361**, 795 (2000) [astro-ph/0004121].
7. R. Prix, G. L. Comer and N. Andersson, "Slowly rotating superfluid Newtonian star model with entrainment," *Astron. & Astrophys.* **381**, 178 (2002) [astro-ph/0107176].
8. R. Prix "Variational derivation of Newtonian multi-fluid hydrodynamics," preprint (Maths Dept., Univ. Southampton, 2002) [physics/0209024].
9. A. F. Andreev and E. P. Bashkin, *Sov. Phys. JETP* **42**, 164 (1975).
10. B. Carter and I. M. Khalatnikov, *Ann. Phys.* **219**, 243 (1992).
11. H. P. Kunzle, *Canad. J. Phys.* **64**, 185 (1986).
12. C. Duval, G. Gibbons and P. Horvathy, *Phys. Rev.* **D43**, 3907 (1991).
13. W. Unruh, *Phys. Rev.* **D51**, 2827 (1995) [gr-qc/9409008].
14. E. A. Milne, *Quart. J. Math.* (*Oxford Ser.*) **5**, 64 (1934).
15. J. J. Moreau, *C.R. Acad. Sci.* (*Paris*) **252**, 2810 (1961).
16. H. K. Moffat, *J. Fluid Mech.* **35**, 117 (1969).
17. B. Gaffet, *J. Fluid Mech.* **156**, 141 (1985).
18. L. Landau and E. Lifchitz, *Physique Théorique, Mécanique des Fluides* (Editions Mir, 1989).
19. B. Carter and N. Chamel, "Covariant analysis of Newtonian multi-fluid models for neutron stars: II Stress-energy tensors and virial theorems," preprint [astro-ph/0312414].