International Journal of Modern Physics D
Vol. 14, No. 5 (2005) 749–774

World Scientific
www.worldscientific.com

COVARIANT ANALYSIS OF NEWTONIAN MULTI-FLUID MODELS FOR NEUTRON STARS: III TRANSVECTIVE, VISCOUS, AND SUPERFLUID DRAG DISSIPATION

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Received 18 December 2003
Revised 15 June 2004
Communicated by P. Laguna

As a follow up to papers dealing firstly with a convective variational formulation in a Milne–Cartan framework for non-dissipative multi-fluid models, and secondly with various ensuing stress energy conservation laws and generalized virial theorems, this work continues a series showing how analytical procedures developed in the context of General Relativity can be usefully adapted for implementation in a purely Newtonian framework where they provide physical insights that are not so easy to obtain by the traditional approach based on a $3 + 1$ space time decomposition. The present paper describes the 4-dimensionally covariant treatment of various dissipative mechanisms, including viscosity in non-superfluid constituents, superfluid vortex drag, ordinary resistivity (mutual friction) between relatively moving non-superfluid constituents, and the transvective dissipation that occurs when matter is transformed from one constituent to another due to chemical disequilibrium such as may be produced by meridional circulation in neutron stars. The corresponding non-dissipative limit cases of vortex pinning, convection and chemical equilibrium are also considered.

Keywords: Newton–Cartan; hydrodynamics; dissipative fluids; superfluid drag; mutual friction; viscous forces; transvective forces.

1. Introduction

This article continues the development[1,2] of a coherent fully covariant approach to the construction and application of Newtonian fluid models of the more general kind required in the context of neutron star phenomena in cases for which it is necessary to allow for independent motion of neutronic and protonic constituents. Generalizing the approach that was originally introduced[3] for the special case of Landau's two-constituent superfluid model, the preceding papers dealt with idealized perfectly conservative models, for which a strictly variational formulation is available. In a complementary treatment by Prix,[4] it has been shown how this covariant variational formulation can be translated into terms of the familiar kind

of 3+1 direct product structure of space with time that has traditionally been used in a non-relativistic approach, and also how it can be extended to allow for dissipative effects — such as chemical reactions and mutual resistivity between relatively moving currents of thermal and other kinds — while retaining as much as possible the convenient[5,6] mathematical machinery provided by the variational approach. In the same spirit, but continuing within a fully covariant framework, the purpose of the present paper is to extend our treatment to allow for a wider range of dissipative mechanisms of the kind[7] likely to be relevant in neutron stars, particularly those due to the presence of superfluid vortices.

In the non-dissipative applications considered in the preceding articles[1,2] the status of any entropy current 4-vector s^μ that may have been involved was effectively the same as that of the other relevant conserved currents with 4-vectors $n_{\rm x}^{\ \mu} = n_{\rm x} u_{\rm x}^{\ \mu}$ for corresponding number density $n_{\rm x}^{\ \mu}$ and unit flow vectors $u_{\rm x}^{\ \mu}$ designated by various values of the chemical index label x. However, in the dissipative applications to be considered here, the entropy density s and the (no longer conserved) entropy current

$$s^\mu = s u_\emptyset^{\ \mu} \tag{1}$$

will have a privileged role, characterizing a corresponding local *thermal rest frame* specified by a unit 4-vector $u_\emptyset^{\ \mu}$ for which the (barred) zero value, $\mathrm{x} = \emptyset$, of the chemical index will be reserved, i.e. we shall set $n_\emptyset = s$, $n_\emptyset^{\ \mu} = s^\mu$.

While the other (particle) currents may still either be conserved, in the sense of having $\nabla_\mu n_{\rm x}^{\ \mu} = 0$ for certain values of the chemical index x, or else may have divergence $\nabla_\mu n_{\rm x}^{\ \mu}$ of unrestricted, positive or negative, sign — corresponding to the possibilities of particle creation or destruction — the second law of thermodynamics stipulates that entropy should never be destroyed, which means that in an entirely self contained treatment we must always have

$$\nabla_\mu s^{\ \mu} \geq 0\,, \tag{2}$$

i.e. $\nabla_\mu n_\emptyset^{\ \mu}$ can never be negative. In some contexts[8] it may however be convenient to work in terms of an open (i.e. not completely self contained) model in which, although not actually destroyed, entropy is nevertheless effectively lost from the system by some local heat removal mechanism — such as the URCA (neutrino–antineutrino pair creation) process in a neutron star core — in which case the relevant remaining entropy current s^μ would not necessarily have to respect the restriction (2), but would be subjected to the modified inequality (23) that is given below.

In the preceding paper[2] it was shown how, in a system governed by a multifluid action variation principle, the relevant (kinetic, internal, and gravitational) Lagrangian action contributions give rise to corresponding variationally defined stress energy contributions that combine to give a total $T_{\rm tot}{}_\nu^{\ \mu}$ which satisfies a Noether type identity of the form

$$\nabla_\mu T_{\rm tot}{}_\nu^{\ \mu} = \sum_{\rm x} f_\mu^{\rm x}\,, \tag{3}$$

in which, for each value of the chemical index x, the 4-covector f^{x}_{μ} denotes the variationally defined non-gravitational force density acting on the corresponding constituent. The strictly conservative case considered in the preceding work was characterized by dynamical equations given, according to the variation principle, just by the requirement that each of the separate force densities f^{x}_{μ} should vanish. The purpose of the present paper is to extend the analysis to a more general category of dynamical equations, whereby the force densities are not required to vanish but are given by non-conservative contributions from dissipative mechanisms of three different kinds, namely as viscosity, resistance against relative motion and transfusion between the various chemical constituents.

Although the suspension of the variation principle leaves a considerable amount of latitude in the way the various kinds of dissipative force may be specified, the admissible forms of force law are considerably restricted by the requirement of compatibility with the second law of thermodynamics as embodied, for a self contained system, in the inequality (2). As in the preceding papers[1,2] our work will be guided by previous experience[9–11] with analogous dissipative effects in a General Relativistic framework, which (contrary to what is commonly supposed) is actually simpler for many purposes, and particularly for the treatment of electromagnetic effects, which are not included (except as possible external background forces) in the present strictly Newtonian analysis.

2. Viscous Stress

The first of the dissipative mechanisms that we need to consider — and the only one that will occur in a single constituent fluid model — is that of *viscosity*, whose effect will be interpretable in terms of a gross stress energy density tensor

$$T_{\mathrm{gro}}{}^{\mu}{}_{\nu} = T_{\mathrm{tot}}{}^{\mu}{}_{\nu} + \sum_{\mathrm{x}} \tau^{\mathrm{x}}_{\nu}{}^{\mu} \,, \tag{4}$$

in which the total $T_{\mathrm{tot}}{}^{\mu}{}_{\nu}$ provided by the previously considered action contributions[2] is supplemented further by viscous stress contributions $\tau^{\mathrm{x}}_{\nu}{}^{\mu}$ that are not obtained from the Lagrangian action but that are included to allow for deviations (from what would otherwise be a local thermal equilibrium state) due to space gradients of the corresponding flow vectors u^{μ}_{x}. In accordance with what is suggested by detailed microscopic analysis of dilute gas models[12] it will be assumed that each such contribution has contravariant components

$$\tau^{\mathrm{x}\,\mu\nu} = \gamma^{\mu\rho}\tau^{\mathrm{x}}_{\rho}{}^{\nu} \tag{5}$$

that are symmetric and purely spacelike, i.e.

$$\tau^{\mathrm{x}\,\mu\nu} = \tau^{\mathrm{x}\,\nu\mu} \,, \qquad \tau^{\mathrm{x}\,\mu\nu} t_{\nu} = 0 \,, \tag{6}$$

(where, as discussed in the preceding work,[1] $\gamma^{\mu\rho}$ is the degenerate Newtonian space metric while t_{μ} is the preferred Newtonian time gradient) and it will also be assumed

that the mixed version is strictly spacelike with respect to the corresponding fluid rest frame, i.e. for each value of the chemical index x we shall have

$$u_{\mathrm{x}}^{\nu}\tau_{\nu}^{\mathrm{x}\,\mu}=0\,. \tag{7}$$

It follows that it will be expressible in the form

$$\tau_{\nu}^{\mathrm{x}\,\mu}=\gamma_{\mathrm{x}\nu\rho}\tau^{\mathrm{x}\,\rho\mu}\,, \tag{8}$$

where $\gamma_{\mathrm{x}\nu\rho}$ is the positive indefinite (rank 3) space metric tensor that would be determined (in the manner described in the preceding work[1]) by choosing the ether reference vector e^{μ} to coincide with the local flow vector u_{x}^{ν}, i.e. it is given by the defining relations

$$\gamma_{\mathrm{x}\nu\rho}\gamma^{\rho\mu}=\gamma_{\mathrm{x}\nu}^{\mu}\,,\qquad \gamma_{\mathrm{x}\nu\rho}u_{\mathrm{x}}^{\rho}=0\,, \tag{9}$$

with

$$\gamma_{\mathrm{x}\nu}^{\mu}=\delta_{\nu}^{\mu}-u_{\mathrm{x}}^{\mu}t_{\nu}\,. \tag{10}$$

In order to set up an appropriate category of models, we proceed on the basis of the postulate that this gross stress energy tensor should satisfy an energy momentum balance condition of the form

$$\nabla_{\mu}T_{\mathrm{gro}\,\nu}^{\ \ \mu}=\sum_{\mathrm{x}}f_{\mathrm{ext}\,\mu}^{\mathrm{x}}\,, \tag{11}$$

in which the terms on the right will all vanish whenever we are dealing with a strictly self contained system, but in which the possibility of external force density contributions $f_{\mathrm{ext}\,\mu}^{\mathrm{x}}$ is included to allow for cases when we are dealing with an open system involving effects such as neutrino emission or interaction with a long range electromagnetic field whose treatment within a model of the non-relativistic kind studied here is prevented by the incompatibility of the necessary Lorentz and Galilean invariance requirements.

Using the specification (4) in conjunction with the Noether identity (3) we see that the dynamical force balance requirement (11) will be expressible simply as

$$\sum_{\mathrm{x}}\tilde{f}_{\mu}^{\mathrm{x}}=0\,, \tag{12}$$

where, for each constituent with label x, the corresponding amalgamated force contribution is defined by

$$\tilde{f}_{\mu}^{\mathrm{x}}=f_{\mu}^{\mathrm{x}}+\nabla_{\nu}\tau_{\mu}^{\mathrm{x}\,\nu}-f_{\mathrm{ext}\,\mu}^{\mathrm{x}}\,. \tag{13}$$

3. The Thermodynamic Positivity Requirement

It is now to be recalled that, according to our preceding work, each of the ordinary local constituent 4-force densities in the Noether identity (3) will be given as the sum of an acceleration contribution and a (gauge dependent) 4-momentum transfer contribution by the formula

$$f^{\mathrm{x}}_{\mu} = \not{f}^{\mathrm{x}}_{\mu} + \pi^{\mathrm{x}}_{\mu} \nabla_{\nu} n^{\nu}_{\mathrm{x}} \,, \tag{14}$$

in which π^{x}_{μ} is the relevant (gauge dependent) 4-momentum covector that is defined in terms of the corresponding locally defined material 4-momentum covector μ^{x}_{μ} and the background gravitational field potential ϕ by

$$\pi^{\mathrm{x}}_{\mu} = \mu^{\mathrm{x}}_{\mu} - m^{\mathrm{x}} \phi t_{\mu} \,, \tag{15}$$

where m^{x} is the relevant particle rest mass parameter, and the (gauge independent) acceleration contribution is specified — using a cross barred symbol — by

$$\not{f}^{\mathrm{x}}_{\mu} = 2 n^{\nu}_{\mathrm{x}} \nabla_{[\nu} \pi^{\mathrm{x}}{}_{\mu]} \,, \tag{16}$$

which therefore satisfies

$$u^{\mu}_{\mathrm{x}} \not{f}^{\mathrm{x}}_{\mu} = 0 \,. \tag{17}$$

In the particular case of the entropy current labelled by the value $\mathrm{x} = \emptyset$, the relevant particle rest mass vanishes, and (as illustrated by the example of the historic Landau model[3]) the corresponding thermal momentum will be directly identifiable with the local temperature covector that is obtained as the partial derivative of the material Lagrangian density Λ with respect to the entropy current, $\Theta_{\mu} = \partial \Lambda / \partial s^{\mu}$ (for fixed values of the other currents) and from which the temperature in the thermal rest frame is obtainable as

$$\Theta = -u^{\mu}_{\emptyset} \Theta_{\mu} \,, \tag{18}$$

i.e. we shall simply have

$$m^{\emptyset} = 0 \,, \qquad \pi^{\emptyset}_{\mu} = \Theta_{\mu} \,. \tag{19}$$

This means that, according to (15) the corresponding thermal 4-force density covector will be given just by

$$f^{\emptyset}_{\mu} = 2 s^{\nu} \nabla_{[\nu} \Theta_{\mu]} + \Theta_{\mu} \nabla_{\nu} s^{\nu} \,, \tag{20}$$

with the implication that its time component in the thermal rest frame will be given simply by

$$u^{\nu}_{\emptyset} f^{\emptyset}_{\nu} = -\Theta \nabla_{\nu} s^{\nu} \,, \tag{21}$$

which will be negative (since the temperature Θ must always be positive) by the second law requirement (2) in any system that is closed in the strong sense of being fully self contained, so that the external forces $f^{\mathrm{x}}_{\mathrm{ext}\,\mu}$ on the right of (11) will vanish.

In the more general case of an open system, in which there may be heat (i.e. thermal energy) loss at a rate (per unit volume in the thermal rest frame) given by

$$\mathcal{Q} = u^{\nu}_{\emptyset} f^{\emptyset}_{\text{ext}\,\nu}\,, \tag{22}$$

due to mechanisms such as neutrino emission, the local formulation of the second law of thermodynamics will no longer take simple form (2) but will be given by the more ubiquitously valid condition

$$\mathcal{Q} + \Theta \nabla_{\nu} s^{\nu} \geq 0\,. \tag{23}$$

4. Transfusive Dissipation

We use the term transfusion to designate processes by which particles of different species are converted into each other by various chemical or nuclear reaction processes, which we shall distinguish by capital Greek letters. The elementary process in some such reaction, with label Ξ say, will involve the creation of a number N^{Ξ}_{A} say of particles of the species with label A, using the convention that the early Latin capital A ranges over the same values as the late Latin index X except for the zero value reserved for the entropy, i.e. it is subject to the restriction $\text{A} \neq \emptyset$ (a simple example of such a process is the decomposition of a Helium nucleus into a pair of neutrons and a pair of protons, so that if we attribute the labels $1, 2, 3$ to neutrons, protons, and Helium nuclei respectively, this reaction will be characterized by $N^{\Xi}_{1} = 2$, $N^{\Xi}_{2} = 2$, $N^{\Xi}_{3} = -1$). In any such reaction, the relevant particle creation numbers are restricted to satisfy the Newtonian mass conservation condition given by

$$\sum_{\text{A}} N^{\Xi}_{\text{A}} m^{\text{A}} = 0\,. \tag{24}$$

By summing over the rates r_{Ξ} of the relevant reactions, the ensuing particle creation rates are obtainable as

$$\nabla_{\nu} n^{\nu}_{\text{A}} = \sum_{\Xi} r_{\Xi} N^{\Xi}_{\text{A}}\,, \tag{25}$$

subject to the proviso that we are dealing with a system that is closed in the weak sense,[13] meaning that there are no external losses or gains of the relevant particle species. Closure in the strong sense, meaning the condition that the model be entirely isolated in the sense of being fully self contained, would imply that all the external 4-force density covectors $f^{\text{X}}_{\text{ext}\,\mu}$ in (11) should vanish, whereas the weak closure condition adopted here corresponds merely to the requirement that the relevant material rest frame components should vanish, i.e.

$$f^{\text{A}}_{\text{ext}\,\nu} u^{\nu}_{\text{A}} = 0\,. \tag{26}$$

This restriction does not exclude the possibility of the kind of external force that might be exerted by interaction with a magnetic field, and since it does not apply

to the special index value $\mathrm{X} = \emptyset$ labelling the entropy, it is also consistent with the possibility of a positive value $\mathcal{Q} > 0$ of the heat loss rate (22) due to a mechanism (such as the URCA process) of the kind that would necessitate replacement of the simple version (2) of the second law of thermodynamics by the more generally applicable version (23).

In accordance with traditional usage[13] in physical chemistry, it is convenient to work with a quantity of the kind for which De Donder introduced the term affinity, according to a specification of the form

$$\mathcal{A}^{\Xi}_{\{\epsilon\}} = -\sum_{\mathrm{A}\neq\emptyset} N^{\Xi}_{\mathrm{A}} \mathcal{E}^{\mathrm{A}}_{\{\epsilon\}} \,, \tag{27}$$

in which the quantities $\mathcal{E}^{\mathrm{A}}_{\{\epsilon\}}$ are the relevant energies per particle of the various species involved, so that the affinity measures the net energy release in an elementary process of the kind (characterized by the label Ξ) under consideration.

A local static chemical (or nuclear) equilibrium state is one in which the relevant affinities vanish. When the deviations from such a state are not too large, it is natural to presume that the reaction rates will be linearly dependent on the affinities according to a prescription of the form

$$r_{\Xi} = \sum_{\Psi} \kappa_{\Xi\Psi} \mathcal{A}^{\Psi}_{\{\epsilon\}} \,, \tag{28}$$

in which the coefficients $\kappa_{\Xi\Psi}$ form a symmetric matrix that must be positive in order to ensure the positivity of the energy release rate $\sum r_{\Xi} \mathcal{A}^{\Xi}_{\{\epsilon\}}$. This implies that the particle creation rates will be given by the formula

$$\nabla_{\nu} n^{\nu}_{\mathrm{A}} = -\sum_{\mathrm{B}} \Xi_{\mathrm{AB}} \mathcal{E}^{\mathrm{B}}_{\{\epsilon\}} \,, \tag{29}$$

in which Ξ_{AB} is a positive indefinite matrix given by

$$\Xi_{\mathrm{AB}} = \sum_{\Xi,\,\Psi} N^{\Xi}_{\mathrm{A}} \kappa_{\Xi\Psi} N^{\Psi}_{\mathrm{B}} \,. \tag{30}$$

This matrix is not positive definite, but only positive indefinite, because it evidently has a null eigenvector provided by the set of particle masses, which will satisfy the condition $\sum_{\mathrm{B}} \Xi_{\mathrm{AB}} m^{\mathrm{B}} = 0$ by the mass conservation law (24). Other such null eigenvectors will be provided by other relevant charge number conservation laws. (The ordinary mass conservation law is interpretable as the Newtonian limit of what is given in a relativistic theory by the baryon conservation law.)

The meaning of the foregoing reasoning is unambiguous under conditions of the kind most commonly considered in physical chemistry in which there are no significant relative motions of the various constituents. However in the more general circumstances we wish to deal with here the meaning of the prescription (27) will be affected by the choice of reference systems in specification of the energies $\mathcal{E}^{\mathrm{A}}_{\{\epsilon\}}$. The most obvious possibility is to evaluate the energy in the thermal rest frame, with respect to which the energy per particle will be given simply by

$$\mathcal{E}^{\mathrm{A}}_{\emptyset} = -\pi^{\mathrm{A}}_{\nu} u^{\nu}_{\emptyset} \,. \tag{31}$$

Such a specification does of course depend on the ether frame used for the definition of the 4-momentum: it can be seen from the analysis of the preceding paper[1] that under a transformation characterized by a Galilean boost velocity vector $b^\mu = \gamma^{\mu\nu}\nabla_\nu\beta$ it will be subjected to a change given by the rule

$$\mathcal{E}^{\mathrm{A}}_{\emptyset} \mapsto \breve{\mathcal{E}}^{\mathrm{A}}_{\emptyset} = \mathcal{E}^{\mathrm{A}}_{\emptyset} + m^{\mathrm{A}} u^{\nu}_{\emptyset} \nabla_\nu \beta - \tfrac{1}{2} m^{\mathrm{A}} b^2 \,. \tag{32}$$

However the mass conservation condition (24) can be seen to ensure that this gauge dependence will cancel out in the corresponding thermal affinity, which is given by

$$\mathcal{A}^{\Xi}_{\emptyset} = -\sum_{\mathrm{A}\neq\emptyset} N^{\Xi}_{\mathrm{A}} \mathcal{E}^{\mathrm{A}}_{\emptyset} = \sum_{\mathrm{A}\neq\emptyset} N^{\Xi}_{\mathrm{A}} \pi^{\mathrm{A}}_{\nu} u^{\nu}_{\emptyset} = \breve{\mathcal{A}}^{\Xi}_{\emptyset} \,. \tag{33}$$

Although it is independent of the choice of ether frame, the specification (33) of the thermal affinity $\mathcal{A}^{\Xi}_{\emptyset}$ will still be indeterminate in applications for which there is no well-defined thermal reference frame, a disadvantage that does not apply to what may be termed the natural affinity, which is given in terms of the corresponding natural energies $\chi^{\mathrm{A}}_{\natural}$ by

$$\mathcal{A}^{\Xi}_{\natural} = -\sum_{\mathrm{A}\neq\emptyset} N^{\Xi}_{\mathrm{A}} \chi^{\mathrm{A}}_{\natural} = \breve{\mathcal{A}}^{\Xi}_{\natural} \,. \tag{34}$$

The natural energy is specified for each constituent as the corresponding chemical potential as measured with respect to its own local rest frame, which means that it will be given by

$$\chi^{\mathrm{A}}_{\natural} = -u^{\mu}_{\mathrm{A}} \chi^{\mathrm{A}}_{\mu} = -u^{\mu}_{\mathrm{A}} \pi^{\mathrm{A}}_{\mu} + \tfrac{1}{2} m^{\mathrm{A}} v^2_{\mathrm{A}} - m^{\mathrm{A}} \phi \,, \tag{35}$$

in which the gauge dependence of the three separate terms on the right cancels out, to give

$$\breve{\chi}^{\mathrm{A}}_{\natural} = \chi^{\mathrm{A}}_{\natural} \,. \tag{36}$$

In the absence, at this stage, of any clear idea of which, if any of these two (thermal and natural) alternatives may be most appropriate for general purposes, we shall proceed in terms of a compromise using a mixed energy $\mathcal{E}^{\mathrm{A}}_{\{\epsilon\}}$ that is defined in terms of a parameter ϵ by

$$\mathcal{E}^{\mathrm{A}}_{\{\epsilon\}} = (1-\epsilon)\mathcal{E}^{\mathrm{A}}_{\emptyset} + \epsilon \chi^{\mathrm{A}}_{\natural} \,, \tag{37}$$

which means that it will transform according to the rule

$$\mathcal{E}^{\mathrm{A}}_{\{\epsilon\}} \mapsto \breve{\mathcal{E}}^{\mathrm{A}}_{\{\epsilon\}} = \mathcal{E}^{\mathrm{A}}_{\{\epsilon\}} + m^{\mathrm{A}}(1-\epsilon)\big(u^{\nu}_{\emptyset} \nabla_\nu \beta - \tfrac{1}{2} b^2\big) \,, \tag{38}$$

so that as before, in consequence of the mass conservation condition (24), the corresponding affinity (25) will be invariant,

$$\breve{\mathcal{A}}^{\Xi}_{\{\epsilon\}} = \mathcal{A}^{\Xi}_{\{\epsilon\}} \,. \tag{39}$$

In this parametrized weighting scheme the special thermal and natural cases are given respectively by

$$\mathcal{A}^{\Xi}_{\{0\}} = \mathcal{A}^{\Xi}_{\emptyset} \,, \qquad \mathcal{A}^{\Xi}_{\{1\}} = \mathcal{A}^{\Xi}_{\natural} \,, \tag{40}$$

and for the general case we shall have

$$\mathcal{A}^{\Xi}_{\{\epsilon\}} = (1-\epsilon)\mathcal{A}^{\Xi}_{\emptyset} + \epsilon\,\mathcal{A}^{\Xi}_{\natural}\,. \tag{41}$$

When the chemical reaction rates and the relative velocities are sufficiently small it will not matter what value is chosen for the parameter ϵ, a consideration that presumably accounts for the lack of attention to this issue in the standard literature on non-equilibrium thermodynamics.[13] However if the chemical reaction rates or the relative velocities are too large, the distinction may become important and in such a case the question of what would be most physically realistic would ultimately need to be decided on the basis of a microscopic analysis of a kind that does not yet seems to have been sufficiently developed. It might turn out that for an accurate out of equilibrium thermodynamic description it would be most appropriate to use something more complicated than the kind of weighted mean adopted here. It is to be remarked that an affinity of the natural kind, as characterized in our present notation by the weighting ansatz $\epsilon = 1$, is what was implicitly used in earlier relativistic work,[9,10] whereas use of an affinity of the thermal kind, as characterized by $\epsilon = 0$, was implicit in more recent and specialized relativistic work.[11] It will be found below that the latter is more satisfactory for applications involving superconductivity.

5. Viscous Dissipation

In a systematic approach to the construction of phenomenological dissipation laws that are consistent with the thermodynamical inequality (23), the first step is to evaluate its left hand side by contraction of the 4-force balance equation (12) with the thermal rest frame unit vector $u^{\nu}_{\emptyset}$ so as to obtain an identity of the form

$$\sum_{\mathrm{X}} u^{\mu}_{\mathrm{X}} f^{\mathrm{X}}_{\mathrm{ext}\,\mu} = \sum_{\mathrm{X}} \left(u^{\mu}_{\emptyset} \pi^{\mathrm{X}}_{\mu} \nabla_{\nu} n^{\nu}_{\mathrm{X}} + u^{\mu}_{\mathrm{X}} \nabla_{\nu} \tau^{\mathrm{X}\,\nu}_{\mu} - v^{\mu}_{\mathrm{X}\emptyset} \tilde{f}^{\mathrm{X}}_{\mu} \right), \tag{42}$$

where $v^{\mu}_{\mathrm{X}\emptyset}$ is the (purely spacelike) velocity difference between the particular unit flow 4-vector u^{μ}_{X} and the thermal rest frame unit 4-vector $u^{\mu}_{\emptyset}$, i.e.

$$v^{\mu}_{\mathrm{X}\emptyset} = u^{\mu}_{\mathrm{X}} - u^{\mu}_{\emptyset}\,, \tag{43}$$

and $\tilde{f}^{\mathrm{X}}_{\mu}$ is the gauge independent force density contribution given by

$$\tilde{f}^{\mathrm{X}}_{\mu} = \tilde{f}^{\mathrm{X}}_{\mu} - \pi^{\mathrm{X}}_{\mu} \nabla_{\nu} n^{\nu}_{\mathrm{X}} = f^{\mathrm{X}}_{\mu} + \nabla_{\nu} \tau^{\mathrm{X}\,\nu}_{\mu} - f^{\mathrm{X}}_{\mathrm{ext}\,\mu}\,. \tag{44}$$

Let us now generalize this specification to a parameter dependent force density contribution given by an expression of the analogous form

$$\tilde{f}^{\mathrm{X}}_{\{\epsilon\}\mu} = f^{\mathrm{X}}_{\{\epsilon\}\mu} + \nabla_{\nu} \tau^{\mathrm{X}\,\nu}_{\mu} - f^{\mathrm{X}}_{\mathrm{ext}\,\mu}\,, \tag{45}$$

in terms of a combination given for $\mathrm{A} \neq \emptyset$, as a function of the weighting parameter ϵ of the preceding section, by the formula

$$f^{\mathrm{A}}_{\{\epsilon\}\mu} = f^{\mathrm{A}}_{\mu} - \left(\pi^{\mathrm{A}}_{\mu} - \epsilon \big(\chi^{\mathrm{A}}_{\mu} + \tfrac{1}{2} m^{\mathrm{A}} v_{\mathrm{A}\emptyset\,\mu} \big) \right) \nabla_{\nu} n^{\nu}_{\mathrm{A}}\,, \tag{46}$$

so that in particular we shall have

$$f^{\mathrm{A}}_{\{0\}\mu} = \not{f}^{\mathrm{A}}_{\mu}\,. \tag{47}$$

We complete the specification for the thermal case with a formula of a rather different form

$$f^{\emptyset}_{\{\epsilon\}\mu} = f^{\emptyset}_{\mu} + \sum_{\mathrm{A}} \Big(\pi^{\mathrm{A}}_{\mu} - \epsilon\big(\chi^{\mathrm{A}}_{\mu} + \tfrac{1}{2}m^{\mathrm{X}}v_{\mathrm{A}\emptyset\,\mu}\big)\Big)\nabla_\nu n^{\ \nu}_{\mathrm{A}}\,, \tag{48}$$

in order to obtain a sum over all constituents that is the same as for the ordinary forces

$$\sum_{\mathrm{X}} f^{\mathrm{A}}_{\{\epsilon\}\mu} = \sum_{\mathrm{X}} f^{\mathrm{X}}_{\mu}\,. \tag{49}$$

This specification has been set up in such a way that, unlike the original (canonically defined) 4-force vectors f^{X}_{μ}, which are frame dependent unless the number currents are separately conserved, the adjusted 4-force densities given by (46) and (48) will have space projected parts that are always unaffected by Galilean (and even Milne) transformations, i.e. they will satisfy the invariance conditions

$$\gamma^{\mu\nu}\breve{f}^{\mathrm{X}}_{\{\epsilon\}\nu} = \gamma^{\mu\nu}f^{\mathrm{X}}_{\{\epsilon\}\nu}\,. \tag{50}$$

The motivation for the introduction of the parametrically adjusted force densities given by the rather elaborately contrived definition (46) is that it enables us to obtain a particularly simple and evocative expression for the entropy term in (42). Using the total mass conservation condition that is obtainable from (24) in the form

$$\sum_{\mathrm{A}\neq\emptyset} m^{\mathrm{A}}\nabla_\nu n^{\ \nu}_{\mathrm{A}} = 0\,, \tag{51}$$

in conjunction with the restrictions (7) and (26) it can be seen to follow from (42) that it will be given — for any chosen value of the weighting parameter ϵ — by

$$\mathcal{Q} + \Theta\nabla_\mu s^\mu = -\sum_{\mathrm{A}} \mathcal{E}^{A}_{\{\epsilon\}}\nabla_\nu n^{\ \nu}_{\mathrm{A}} - \sum_{\mathrm{X}} \tau^{\mathrm{X}}_{\nu}{}^{\mu}\nabla_\mu u^{\ \nu}_{\mathrm{X}} - \sum_{\mathrm{A}} v^{\mu}_{\mathrm{A}\emptyset}\,\tilde{f}^{\mathrm{A}}_{\{\epsilon\}\,\mu}\,. \tag{52}$$

In view of the second law requirement (23) to the effect that the left hand side of (52) should be positive, the choice of admissible dissipation laws will be restricted by the condition that it should be such as to ensure that the sum of the terms on the right of (52) should also be positive. Although many other (generally more complicated) ways involving various kinds of cross coupling are conceivable, it will be adequate for most purposes to do this in the most obvious way by ensuring that each of the three sums on the right of (52) is separately positive.

In so far as the first of these terms is concerned, this desideratum of positivity is already satisfied by the ansatz of the previous section, which — for the chosen value of ϵ — gives an expression of the form

$$-\sum_{\mathrm{A}\neq\emptyset} \mathcal{E}^{\mathrm{A}}_{\{\epsilon\}}\nabla_\nu n^{\ \nu}_{\mathrm{A}} = \sum_{\Xi,\,\Psi} \mathcal{A}^{\Xi}_{\{\epsilon\}}\kappa_{\Xi\Psi}\mathcal{A}^{\Psi}_{\{\epsilon\}}\,, \tag{53}$$

whose positivity is evidently ensured by the condition that the transfusion matrix $\kappa_{\Xi\Psi}$ should be positive definite.

To deal with the second term, we exploit the possibility of rewriting the negative of the contribution of each constituent as

$$\tau_{\nu}^{\mathrm{x}\,\mu}\nabla_{\mu}u_{\mathrm{x}}^{\nu}=\gamma_{\mu\rho}\tau^{\mathrm{x}\,\rho\nu}\gamma_{\nu\sigma}\,\theta_{\mathrm{x}}^{\mu\sigma}\,, \tag{54}$$

where $\theta_{\mathrm{x}}^{\mu\sigma}$ is the symmetric spacelike expansion rate tensor given by

$$\theta_{\mathrm{x}}^{\mu\sigma}=\gamma^{\nu(\mu}\nabla_{\nu}u_{\mathrm{x}}^{\sigma)}\,,\qquad \theta_{\mathrm{x}}^{\mu\sigma}t_{\sigma}=0\,. \tag{55}$$

This tensor is decomposable in a well-defined (Galilean and even Milne gauge independent) manner into tensorially irreducible parts in the form

$$\theta_{\mathrm{x}}^{\mu\nu}=\sigma_{\mathrm{x}}^{\mu\nu}+\tfrac{1}{3}\theta_{\mathrm{x}}\gamma^{\mu\nu}\,, \tag{56}$$

where $\sigma_{\mathrm{x}}^{\mu\nu}$ is the trace free shear rate tensor and θ_{x} is the scalar expansion rate, as characterized by

$$\gamma_{\mu\nu}\sigma^{\mu\nu}=0\,,\qquad \theta_{\mathrm{x}}=\gamma_{\mu\nu}\theta_{\mathrm{x}}^{\mu\nu}\,. \tag{57}$$

This enables us to write the negative of viscosity term in (42) as

$$\sum_{\mathrm{x}}\,\tau_{\nu}^{\mathrm{x}\,\mu}\nabla_{\mu}u_{\mathrm{x}}^{\nu}=\sum_{\mathrm{x}}\,\gamma_{\mu\rho}\gamma_{\nu\sigma}\tau^{\mathrm{x}\,\mu\nu}\sigma_{\mathrm{x}}^{\rho\sigma}+\tfrac{1}{3}\sum_{\mathrm{x}}\,\gamma_{\mu\nu}\tau^{\mathrm{x}\,\mu\nu}\theta_{\mathrm{x}}\,. \tag{58}$$

In order to ensure that this total is negative as required, the obvious generalization of the ansatz that is familiar in the case of a single constituent fluid is to postulate that the relevant stress contributions are given by an expression of the form

$$\tau^{\mathrm{x}\,\mu\nu}=-2\sum_{\mathrm{Y}}\,\eta^{\mathrm{xy}}\sigma_{\mathrm{y}}^{\mu\nu}-\sum_{\mathrm{Y}}\,\zeta^{\mathrm{xy}}\theta_{\mathrm{y}}\gamma^{\mu\nu}\,, \tag{59}$$

where η^{xy} is a positive definite or indefinite but in any case (by the Onsager principle) symmetric matrix of shear viscosity coefficients, and ζ^{xy} is a similarly symmetric positive definite or indefinite matrix of bulk viscosity coefficients.

6. Ordinary Resistive Dissipation

To complete the determination of the dynamical equations of motion it remains to specify the space components of the force on each constituent. The most obvious way of doing this in such a way as to ensure consistency with the total force balance condition (12) is to take them to consist of sums of pairwise interaction contributions in the form

$$\gamma^{\mu\nu}\,\tilde{f}^{\mathrm{x}}_{\{\epsilon\}\,\nu}=\sum_{\mathrm{Y}}\,f^{\mathrm{xy}\,\mu}\,, \tag{60}$$

subject to the conditions

$$f^{\mathrm{xy}\,\mu}=-f^{\mathrm{yx}\,\mu}\,,\qquad t_{\mu}f^{\mathrm{xy}\,\mu}=0\,. \tag{61}$$

It is possible to conceive situations in which a more elaborate construction procedure might be needed, but we shall not envisage such complications here.

Proceeding on the basis of the ansatz (60) we must now consider how the admissible forms of the two-constituent interaction force densities $f^{\mathrm{XY}\,\mu}$ are restricted by the second law of thermodynamics. As we have already chosen rules that ensure the positivity of the first two terms on the right of (52), this restriction will amount just to the requirement of positivity of the final term, which will be given by

$$-\sum_{\mathrm{X}} v^{\mu}_{\mathrm{X}\emptyset}\tilde{f}^{\mathrm{X}}_{\{\epsilon\}\,\mu}=\sum_{\mathrm{X},\mathrm{Y}} v_{\mathrm{X}\emptyset\,\mu}f^{\mathrm{YX}\,\mu}=\tfrac{1}{2}\sum_{\mathrm{X},\mathrm{Y}} v_{\mathrm{XY}\,\mu}f^{\mathrm{YX}\,\mu}\,. \tag{62}$$

in which the (gauge invariant) relative velocity v^{μ}_{XY} and the corresponding (gauge dependent) covector $v_{\mathrm{XY}\,\mu}$ are defined by

$$v^{\mu}_{\mathrm{XY}}=v^{\mu}_{\mathrm{X}\emptyset}-v^{\mu}_{\mathrm{Y}\emptyset}=u^{\mu}_{\mathrm{X}}-u^{\mu}_{\mathrm{Y}}\,,\qquad v_{\mathrm{XY}\,\mu}=\gamma_{\mu\nu}v^{\nu}_{\mathrm{XY}}\,. \tag{63}$$

The obvious way to fulfil this requirement is to suppose that the forces are due just to resistivity of the ordinary kind, which means that for each pair of distinct constituent label values $\neq \mathrm{Y}$ the corresponding contribution will be given by an ordinary positive resistivity coefficient $Z^{\mathrm{XY}}=Z^{\mathrm{YX}}\geq 0$ according to the specification

$$f^{\mathrm{YX}\,\mu}=Z^{\mathrm{XY}}v^{\mu}_{\mathrm{XY}}\,. \tag{64}$$

It follows that the third term on the right of (52) will be given by

$$-\sum_{\mathrm{A}} v^{\mu}_{\mathrm{A}\emptyset}\,\tilde{f}^{\mathrm{A}}_{\{\epsilon\}\,\mu}=\tfrac{1}{2}\sum_{\mathrm{X},\mathrm{Y}} Z^{\mathrm{XY}}v^{\mu}_{\mathrm{XY}}\gamma_{\mu\nu}v^{\nu}_{\mathrm{XY}}\,, \tag{65}$$

which shows that it does indeed satisfy the required positivity condition.

Having thus obtained an appropriate resistivity formula (60) for the gauge invariant force density components $\gamma^{\mu\nu}\tilde{f}^{\mathrm{X}}_{\{\epsilon\}\,\nu}$ we can immediately use the defining relation

$$\tilde{f}^{\mathrm{A}}_{\mu}=\tilde{f}^{\mathrm{A}}_{\{\epsilon\}\mu}-\Big(\pi^{\mathrm{A}}_{\mu}-\epsilon\big(\chi^{\mathrm{A}}_{\mu}+\tfrac{1}{2}m^{\mathrm{A}}v_{\mathrm{A}\emptyset\,\mu}\big)\Big)\nabla_{\nu}n^{\nu}_{\mathrm{A}}\,, \tag{66}$$

to provide a corresponding formula for the original unadjusted (gauge dependent) material force density components, which will be given by

$$\gamma^{\mu\nu}\tilde{f}^{\mathrm{A}}_{\nu}=\sum_{\mathrm{Y}} Z^{\mathrm{AY}}v^{\mu}_{\mathrm{YA}}-\Big(\gamma^{\mu\nu}\pi^{\mathrm{A}}_{\nu}-\epsilon\Big(\gamma^{\mu\nu}\chi^{\mathrm{A}}_{\nu}+\tfrac{1}{2}m^{\mathrm{A}}v^{\ \mu}_{\mathrm{A}\emptyset}\Big)\Big)\nabla_{\nu}n^{\nu}_{\mathrm{A}}\,. \tag{67}$$

7. Superfluid Drag Dissipation

The simple kind of resistive dissipation mechanism described in the previous section will not be operational in the case of a constituent that is superfluid. To deal with such cases, let us use indices I, J that range over the values (if any) of X referring to a superfluid constituent, while using indices C, D that range over the values of X referring to the remaining — normal — constituents.

On a sufficiently small — mesoscopic — scale, superfluidity can be dealt with directly in terms of the kind of model set up in the preceding section by imposing the following conditions, of which the first and most obvious is simply the requirement that the relevant superfluid viscosity and resistivity coefficients should vanish, i.e.

$$\eta^{\text{IX}} = 0\,, \qquad \zeta^{\text{IX}}\,, \qquad Z^{\text{IX}} = 0\,, \tag{68}$$

which implies, by (59) that the corresponding viscous tension contributions $\tau^{\text{I}\,\mu\nu}$ will vanish. The next condition is that the superfluid constituents are not directly subject to any external force,

$$f^{\text{I}}_{\text{ext}\,\mu} = 0\,, \tag{69}$$

so that we shall be able to make the identifications

$$\tilde{f}^{\text{I}}_{\mu} = f^{\text{I}}_{\mu}\,. \tag{70}$$

The final condition that needs to be imposed is that we should be able to make the identification

$$\tilde{f}^{\text{I}}_{\{\epsilon\}\,\mu} = f^{\text{I}}_{\{0\}\,\mu}\,, \tag{71}$$

where (by definition)

$$f^{\text{I}}_{\{0\}\,\mu} = n_{\text{I}}{}^{\nu}\varpi^{\text{I}}_{\nu\mu}\,, \tag{72}$$

which will follow from (70) subject to the requirement that we adopt the thermal affinity ansatz,

$$\epsilon = 0\,, \tag{73}$$

or else that the relevant chemical reaction rates are set to zero so that the superfluid particle creation rates $\nabla_\nu n_{\text{I}}{}^{\nu}$ all vanish, in which case the value of ϵ does not matter. When the conditions (68), (69) and (73) are all satisfied it can be seen that the equations of motion set up in the preceding sections will be consistent with the restraint to the effect that the relevant superfluid vorticities

$$\varpi^{\text{I}}_{\mu\nu} = 2\nabla_{[\mu}\pi^{\text{I}}_{\nu]}\,, \tag{74}$$

should vanish, as necessary for the existence of corresponding local superfluid phase scalars φ^{I} such that $N_{\text{I}}\pi^{\text{I}}_{\nu} = \hbar\nabla_\nu\varphi^{\text{I}}$, where N_{I} is the number (2 for the usual case of a Cooper type pair) of constituent particle in a boson of the superfluid condensate under consideration.

For application on larger scales such a mesoscopic description is inadequate, and must be replaced by a macroscopic description that allows non-vanishing superfluid vorticities $\varpi^{\text{I}}_{\mu\nu}$, which are interpretable as representing the average effect of a fibration by microscopic vortex tubes on which the mesoscopic irrotationality condition breaks down. This interpretation means that although it does not have to vanish, the macroscopic vorticity 2-form of a superfluid constituent cannot have an arbitrary algebraic form but must satisfy the degeneracy condition

$$\varpi^{\text{I}}_{[\mu\nu}\varpi^{\text{I}}_{\rho]\sigma} = 0\,, \tag{75}$$

(which means that its matrix rank is not four as in the generic case but two, since its antisymmetry excludes the possibility of an odd valued rank) in order for the null eigenvectors that generate the vorticity flux two surfaces to exist. As remarked in the context of the analogous relativistic problem,[11] the only obvious way of setting up a force law that satisfies this condition is to postulate that it should have the form

$$f^{\mathrm{I}}_{\{0\}\,\mu} = n_{\mathrm{I}} \varpi^{\mathrm{I}}_{\mu\nu} V_{\mathrm{I}}{}^{\nu} \,, \tag{76}$$

in which $V_{\mathrm{I}}{}^{\nu}$ is some vector such that the combination $u_{\mathrm{I}}{}^{\nu} + V_{\mathrm{I}}{}^{\nu}$ is one of the null eigenvectors generating the flux surfaces of the vorticity flux $\varpi^{\mathrm{I}}_{\mu\nu}$, i.e. such that $\varpi^{\mathrm{I}}_{\mu\nu}(u_{\mathrm{I}}{}^{\nu} + V_{\mathrm{I}}{}^{\nu}) = 0$.

The superfluidity ansatz (76) can be applied within the framework set up in the preceding section by supposing that $V_{\mathrm{I}}{}^{\nu}$ is decomposable as a sum of separate contributions $V_{\mathrm{I}}{}^{\mathrm{C}\,\nu}$ from the various non-superfluid contributions, in the form

$$V_{\mathrm{I}}{}^{\nu} = \sum_{\mathrm{C}} V_{\mathrm{I}}{}^{\mathrm{C}\,\nu} \,, \tag{77}$$

so that the corresponding space projected force contributions in the decomposition (60) can be taken to be given by

$$f^{\mathrm{IC}\,\mu} = n_{\mathrm{I}} \gamma^{\mu\nu} \varpi^{\mathrm{I}}_{\nu\rho} V_{\mathrm{I}}{}^{\mathrm{C}\,\rho} \,. \tag{78}$$

As in the preceding section, we now need to find a procedure for choosing the force contributions in such a way as to ensure the positivity of the final term on the right of (52), which will now be given as a combination of normal and superfluid contributions in the form

$$-\sum_{\mathrm{X}} v^{\mu}_{\mathrm{X}\emptyset}\, \tilde{f}^{\mathrm{X}}_{\{0\}\,\mu} = \tfrac{1}{2} \sum_{\mathrm{C,D}} v_{\mathrm{CD}\,\mu} f^{\mathrm{DC}\,\mu} + \sum_{\mathrm{C,I}} v_{\mathrm{CI}\,\mu} f^{\mathrm{IC}\,\mu} \,. \tag{79}$$

As before, we can deal with the normal part by taking the relevant force contributions to be given for $\mathrm{C} \neq \mathrm{D}$ by a set of normal resistivity coefficients $Z^{\mathrm{CD}} = Z^{\mathrm{CD}} \geq 0$ according to the simple specification

$$f^{\mathrm{DC}\,\mu} = Z^{\mathrm{CD}} v^{\mu}_{\mathrm{CD}} \,, \tag{80}$$

which will automatically take care of the positivity of the first term on the right of (79), but the superfluid contributions $f^{\mathrm{IC}\,\mu}$ will need to be handled in a different manner.

The remaining positivity requirement that still needs to be satisfied is that of the second term on the right of (79), which can be rewritten as

$$\sum_{\mathrm{C,I}} v_{\mathrm{CI}\,\mu} f^{\mathrm{IC}\,\mu} = \sum_{\mathrm{C,I}} v^{\mu}_{\mathrm{CI}}\, n_{\mathrm{I}} \varpi^{\mathrm{I}}_{\mu\nu} V_{\mathrm{I}}{}^{\mathrm{C}\,\nu} = \sum_{\mathrm{C,I}} u^{\mu}_{\mathrm{C}}\, n_{\mathrm{I}} \varpi^{\mathrm{I}}_{\mu\nu} V_{\mathrm{I}}{}^{\mathrm{C}\,\nu} \,, \tag{81}$$

where the last step is obtained by substituting (77) in (76) and using the identity $u_{\mathrm{I}}{}^{\mu} f^{\mathrm{I}}_{\{0\}\,\mu} = 0$. We now proceed in a manner analogous to that by which the ordinary resistivity forces were introduced above, which means that we ensure the positivity

of the separate terms in the sum on the right of (81) by taking each vector $V_{\mathrm{I}}^{\mathrm{C}\,\mu}$ to be given by an expression of the form

$$V_{\mathrm{I}}^{\mathrm{C}\,\mu} = -c_{\mathrm{I}}^{\mathrm{C}}(w^{\mathrm{I}})^{-1}\gamma^{\mu\nu}\varpi^{\mathrm{I}}_{\nu\rho}u_{\mathrm{C}}^{\rho}\,, \tag{82}$$

where $c_{\mathrm{I}}^{\mathrm{C}}$ is a positive drag coefficient and we have included a positive normalization factor given by the magnitude $w^{\mathrm{I}} \geq 0$ of the (gauge independent) space projected vorticity vector

$$w^{\mathrm{I}\,\mu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\varpi^{\mathrm{I}}_{\nu\rho}\,, \tag{83}$$

according to the specification

$$(w^{\mathrm{I}})^2 = \tfrac{1}{2}\gamma^{\mu\nu}\gamma^{\rho\sigma}\varpi^{\mathrm{I}}_{\mu\rho}\varpi^{\mathrm{I}}_{\nu\sigma} = \gamma_{\mu\nu}w^{\mathrm{I}\,\mu}w^{\mathrm{I}\,\nu}\,. \tag{84}$$

The prescription (84) is evidently equivalent to the adoption of a force law of the form

$$f^{\mathrm{IC}\,\mu} = c_{\mathrm{I}}^{\mathrm{C}}n_{\mathrm{I}}w^{\mathrm{I}}\perp^{\mathrm{I}\mu}_{\nu}u_{\mathrm{C}}^{\nu}\,, \tag{85}$$

where

$$\perp^{\mathrm{I}\mu}_{\nu} = (w^{\mathrm{I}})^{-2}\gamma^{\mu\rho}\gamma^{\sigma\tau}\varpi^{\mathrm{I}}_{\tau\rho}\varpi^{\mathrm{I}}_{\sigma\nu}\,. \tag{86}$$

Adding up the resulting contributions we finally obtain

$$\gamma^{\mu\nu}f^{\mathrm{I}}_{\{0\}\,\nu} = n_{\mathrm{I}}\gamma^{\mu\nu}\varpi^{\mathrm{I}}_{\nu\rho}V_{\mathrm{I}}^{\rho}\,, \tag{87}$$

which (by the identity $u_{\mathrm{I}}^{\nu}f^{\mathrm{I}}_{\{0\}\,\nu} = 0$) automatically provides the required result (76) with

$$V_{\mathrm{I}}^{\mu} = -(w^{\mathrm{I}})^{-1}\gamma^{\mu\nu}\varpi^{\mathrm{I}}_{\nu\rho}\sum_{\mathrm{C}}c_{\mathrm{I}}^{\mathrm{C}}u_{\mathrm{C}}^{\rho}\,. \tag{88}$$

To relate this 4-dimensionally covariant formulation to the traditional Newtonian terminology using a $3+1$ decomposition based on some particular choice of ether frame vector e^{μ}, it is useful to introduce the (frame dependent) vorticity surface generating unit 4-vector $\breve{u}_{\mathrm{I}}^{\mu}$ and its associated 3-velocity vector $\breve{v}_{\mathrm{I}}^{\mu} = \breve{u}_{\mathrm{I}}^{\mu} - e^{\mu}$ by the defining conditions

$$\varpi^{\mathrm{I}}_{\mu\nu}\breve{u}_{\mathrm{I}}^{\nu} = 0\,, \qquad \gamma_{\mu\nu}w^{\mathrm{I}\,\mu}\breve{v}_{\mathrm{I}}^{\nu} = 0\,. \tag{89}$$

In terms of such a vorticity flux velocity vector, the degenerate vorticity 2-form $\varpi^{\mathrm{I}}_{\mu\nu}$ will be expressible as

$$\varpi^{\mathrm{I}}_{\mu\nu} = \big(\varepsilon_{\mu\nu\rho} + 2t_{[\mu}\varepsilon_{\nu]\sigma\rho}\breve{v}_{\mathrm{I}}^{\sigma}\big)w^{\mathrm{I}\,\rho}\,, \tag{90}$$

and the (rank 2) projection tensor in (85) will be given by

$$\perp^{\mathrm{I}\mu}_{\nu} = \gamma^{\mu}_{\nu} - \breve{v}_{\mathrm{I}}^{\mu}t_{\nu} - (w^{\mathrm{I}})^{-2}w^{\mathrm{I}\,\mu}w^{\mathrm{I}}_{\nu}\,. \tag{91}$$

It can thus be seen that the projected velocity on the right of (85) will be given by

$$\perp^{\mathrm{I}\mu}_{\nu}u_{\mathrm{C}}^{\nu} = v_{\mathrm{C}}^{\mu} - (w^{\mathrm{I}})^{-2}w^{\mathrm{I}\,\mu}w^{\mathrm{I}\,\nu}v_{\mathrm{C}\,\nu} - \breve{v}_{\mathrm{I}}^{\mu}\,. \tag{92}$$

8. The Limit Cases of Convection and Pinning

If some resistivity coefficient, Z^{XY}, is very large, the corresponding velocity difference, v^{μ}_{YX} will tend to be very small. In such a case it will often be convenient to use a simplified dynamical treatment based on the relevant *convection ansatz* to the effect that the velocity difference in question should actually vanish. To deal with this kind of convective limit, i.e. a case in which the number of independent normal velocities is smaller that the number of independent chemical constituents, it is convenient to introduce a new kind of index $\langle U\rangle$, distinguished by surrounding angle brackets, to label the independent velocities and the corresponding comoving subsets of constituents that are characterized as equivalence classes by

$$\mathrm{C}\in\langle U\rangle \Leftrightarrow u^{\mu}_{\mathrm{C}}=u^{\mu}_{\langle U\rangle}\,. \tag{93}$$

In particular we shall use the label $\langle\emptyset\rangle$ for the class of constituents that are convected with the entropy, so that

$$\mathrm{C}\in\langle\emptyset\rangle \Leftrightarrow v^{\mu}_{\mathrm{C}\emptyset}=0\,. \tag{94}$$

For each such class of comoving constituents, it will be useful to define combined values of additive quantities such as force density and stress using notation illustrated by the examples

$$\tilde{f}^{\langle U\rangle}_{\mu}=\sum_{\mathrm{C}\in\langle U\rangle}\tilde{f}^{\mathrm{C}}_{\mu}\,,\quad f^{\langle U\rangle}_{\mathrm{ext}\,\mu}=\sum_{\mathrm{C}\in\langle U\rangle}f^{\mathrm{C}}_{\mathrm{ext}\,\mu}\,,\quad \tau^{\langle U\rangle\,\mu}_{\nu}=\sum_{\mathrm{C}\in\langle U\rangle}\tau^{\mathrm{C}\,\mu}_{\nu}\,, \tag{95}$$

and more particularly for the ordinary dynamical 4-force density by

$$f^{\langle U\rangle}_{\mu}=\sum_{\mathrm{C}\in\langle U\rangle}f^{\mathrm{C}}_{\mu}=2u^{\nu}_{\langle U\rangle}\sum_{\mathrm{C}\in\langle U\rangle}n_{\mathrm{C}}\nabla_{[\nu}\pi^{\mathrm{C}}{}_{\mu]}+\sum_{\mathrm{C}\in\langle U\rangle}\pi^{\mathrm{C}}_{\mu}\nabla_{\nu}n^{\nu}_{\mathrm{C}}\,. \tag{96}$$

Using this notation we can regroup the terms on the right of the general entropy creation formula (52) in the form

$$\begin{aligned}\mathcal{Q}+\Theta\nabla_{\mu}s^{\mu}=&-\sum_{\mathrm{A}}\mathcal{E}^{\mathrm{A}}_{\{\emptyset\}}\nabla_{\nu}n^{\nu}_{\mathrm{A}}-\sum_{\langle U\rangle}\tau^{\langle U\rangle\,\mu}_{\nu}\nabla_{\mu}u^{\nu}_{\langle U\rangle}\\&-\sum_{\langle U\rangle}v^{\mu}_{\langle U\rangle\emptyset}\tilde{f}^{\langle U\rangle}_{\{0\}\,\mu}-\sum_{\mathrm{I}}v^{\mu}_{\mathrm{I}\emptyset}f^{\mathrm{I}}_{\{0\}\,\mu}\,.\end{aligned} \tag{97}$$

Using the notation introduced above, according to which

$$\tilde{f}^{\langle U\rangle}_{\{0\}\mu}=f^{\langle U\rangle}_{\{0\}\mu}+\nabla_{\nu}\tau^{\langle U\rangle\,\nu}_{\mu}-f^{\langle U\rangle}_{\mathrm{ext}\,\mu}\,. \tag{98}$$

The first term on the right of (97) is the rate of chemical energy release which we deal with exactly as before in (29) by setting

$$\nabla_{\nu}n^{\nu}_{\mathrm{A}}=-\sum_{\mathrm{B}}\Xi_{AB}\mathcal{E}^{\mathrm{B}}_{\{\emptyset\}}\,, \tag{99}$$

where Ξ_{AB} is the same reactivity matrix as was introduced above. The second term on the right of (97) is the viscous energy dissipation rate which we deal with in the same manner as in (59) by setting

$$\tau^{\langle U\rangle\,\mu\nu}=-2\sum_{\langle V\rangle}\eta^{\langle U\rangle\langle V\rangle}\sigma^{\mu\nu}_{\langle V\rangle}-\sum_{\langle V\rangle}\zeta^{\langle U\rangle\langle V\rangle}\theta_{\langle V\rangle}\gamma^{\mu\nu}\,. \tag{100}$$

The only difference being that the positive shear viscosity coefficients $\eta^{\langle U\rangle\langle V\rangle} = \eta^{\langle V\rangle\langle U\rangle}$ and bulk velocity coefficients $\zeta^{\langle U\rangle\langle V\rangle} = \zeta^{\langle V\rangle\langle U\rangle}$ now only need to be specified for the restricted range of the index $\langle U\rangle$ labelling the comoving equivalence classes rather than for the full range of the normal constituent label C. It will similarly be sufficient to specify just a restricted range of positive resistivity coefficients $Z^{\langle U\rangle\langle V\rangle} = Z^{\langle V\rangle\langle U\rangle}$ for $\langle U\rangle \neq \langle V\rangle$ to specify the resistivity contribution in a force formula whereby the terms in the original expression (62) are regrouped in the form

$$\gamma^{\mu\nu}\,\tilde{f}^{\langle U\rangle}_{\{0\}\,\nu} = \sum_{\langle V\rangle} f^{\langle U\rangle\langle V\rangle\,\mu} + \sum_{\mathrm{I}} f^{\langle U\rangle\mathrm{I}\,\mu}\,, \tag{101}$$

in which the resistive force density terms are given by

$$f^{\langle U\rangle\langle V\rangle\,\mu} = Z^{\langle U\rangle\langle V\rangle}\, v^{\ \mu}_{\langle V\rangle\langle U\rangle}\,. \tag{102}$$

It remains to specify the vortex drag terms $f^{\mathrm{I}\langle U\rangle\,\mu} = -f^{\langle U\rangle\mathrm{I}\,\mu}$, which combine to give the space convected superfluid force densities as

$$\gamma^{\mu\nu}\, f^{\mathrm{I}}_{\{0\}\,\nu} = \sum_{\langle U\rangle} f^{\mathrm{I}\langle U\rangle\,\mu}\,, \tag{103}$$

so that the last two terms in (97) can be recombined in the form

$$\begin{aligned} &-\sum_{\langle U\rangle} v^{\ \mu}_{\langle U\rangle\emptyset}\,\tilde{f}^{\langle U\rangle}_{\{0\}\,\mu} - \sum_{\mathrm{I}} v^{\mu}_{\mathrm{I}\emptyset}\, f^{\mathrm{I}}_{\{0\}\,\mu} \\ &\qquad = \tfrac{1}{2}\sum_{\langle U\rangle,\langle V\rangle} v_{\langle V\rangle\langle U\rangle\,\mu} f^{\langle U\rangle\langle V\rangle\,\mu} + \sum_{\langle U\rangle,\mathrm{I}} v_{\langle U\rangle\mathrm{I}\,\mu} f^{\mathrm{I}\langle U\rangle\,\mu}\,. \end{aligned} \tag{104}$$

According to the reasoning of the preceding section, these contributions should be given by expressions of the form

$$f^{\mathrm{I}\langle U\rangle\,\mu} = n_{\mathrm{I}}\gamma^{\mu\nu}\varpi^{\mathrm{I}}_{\nu\rho}V^{\langle U\rangle\,\rho}_{\mathrm{I}} \tag{105}$$

for a set of generalized velocity vectors that add up to give a sum

$$V^{\mu}_{\mathrm{I}} = \sum_{\langle V\rangle} V^{\langle U\rangle\,\mu}_{\mathrm{I}}\,, \tag{106}$$

in terms of which we shall get

$$\varpi^{\mathrm{I}}_{\mu\nu}\big(u^{\mu}_{\mathrm{I}} + V^{\mu}_{\mathrm{I}}\big) = 0\,, \tag{107}$$

and

$$f^{\mathrm{I}}_{\{0\}\,\mu} = \varpi^{\mathrm{I}}_{\mu\nu}V^{\mu}_{\mathrm{I}}\,. \tag{108}$$

The final drag dissipative term in (104) can thereby be rewritten as

$$\sum_{\langle U\rangle,\mathrm{I}} v_{\langle U\rangle\mathrm{I}\,\mu} f^{\mathrm{I}\langle U\rangle\,\mu} = \sum_{\langle U\rangle,\mathrm{I}} n_{\mathrm{I}} u^{\ \mu}_{\langle U\rangle}\,\varpi^{\mathrm{I}}_{\mu\nu}V^{\langle U\rangle\,\nu}_{\mathrm{I}}\,. \tag{109}$$

By the same reasoning as in the preceding section, we can ensure the required positivity of the total (103) by adoption of an ansatz of the form

$$V^{\langle U\rangle\,\mu}_{\mathrm{I}} = -c^{\langle U\rangle}_{\mathrm{J}}\,(w^{\mathrm{J}})^{-1}\gamma^{\mu\nu}\varpi^{\mathrm{J}}_{\nu\rho}u^{\ \nu}_{\langle U\rangle}\,, \tag{110}$$

which is equivalent to setting

$$f^{\mathrm{J}\langle U\rangle\,\mu} = c_{\mathrm{J}}^{\langle U\rangle} n_{\mathrm{J}} w^{\mathrm{J}} \perp^{\mathrm{J}\mu}_{\nu} u^{\ \nu}_{\langle U\rangle}\,, \tag{111}$$

for a set of positive drag coefficients $c_{\mathrm{J}}^{\langle U\rangle}$ where J, like I, ranges over the set of superfluid index labels.

There is however an extreme limit known as pinning — the analogue for a superfluid constituent of convection in the normal case — representing what will occur if some drag coefficient $c_{\mathrm{H}}^{\langle U\rangle}$ say is very large, in which case the normal flow vector $u^{\ \mu}_{\langle U\rangle}$ will be constrained to lie in the relevant vorticity surface, a requirement that is evidently expressible as the condition

$$\varpi^{\mathrm{H}}_{\mu\nu} u^{\ \nu}_{\langle U\rangle} = 0\,, \tag{112}$$

which is evidently sufficient to ensure that the corresponding dissipation term in the sum (109) will simply vanish.

This can be achieved by taking the term $f^{\mathrm{H}\langle U\rangle\,\mu}$ to be given by the dissipative drag prescription of the form (111) that applies to the other contributions $f^{\mathrm{H}\langle V\rangle\,\mu}$ for $\langle V\rangle \neq \langle U\rangle$ — and to all the corresponding force contributions for the unpinned constituents — but instead by the alternative ansatz

$$V_{\mathrm{H}}^{\langle U\rangle\,\mu} = v^{\ \mu}_{\langle U\rangle\mathrm{H}} - \sum_{\langle V\rangle\neq\langle U\rangle} V_{\mathrm{H}}^{\langle V\rangle\,\mu} \tag{113}$$

that is equivalent to the formula

$$f^{\mathrm{H}\langle U\rangle\,\mu} = n_{\mathrm{H}} \gamma^{\mu\nu} \varpi^{\mathrm{H}}_{\nu\rho} v^{\ \rho}_{\langle U\rangle\mathrm{H}} - \sum_{\langle V\rangle\neq\langle U\rangle} f^{\mathrm{H}\langle V\rangle\,\mu}\,, \tag{114}$$

which is chosen in such a way as to ensure that, according to (103), the total force acting on the pinned constituent will be given by

$$\gamma^{\mu\nu}\, f^{\mathrm{H}}_{\{0\}\,\nu} = n_{\mathrm{H}} \gamma^{\mu\nu} \varpi^{\mathrm{H}}_{\nu\rho} v^{\ \rho}_{\langle U\rangle\mathrm{H}}\,. \tag{115}$$

For any value of the superfluid constituent index I, not just for the pinned index value H we have been considering, knowledge of the contravariant space projection $\gamma^{\mu\nu} f^{\mathrm{I}}_{\{0\}\,\nu}$ will be sufficient for the complete specification of $f^{\mathrm{I}}_{\{0\}\,\nu}$ due to the identity $u^{\mu}_{\mathrm{I}} f^{\mathrm{I}}_{\{0\}\,\mu}$ which implies that we shall have

$$f^{\mathrm{I}}_{\{0\}\,\nu} = \gamma_{\mathrm{I}\,\mu\nu} \gamma^{\nu\rho} f^{\mathrm{I}}_{\{0\}\,\rho} \tag{116}$$

with $\gamma_{\mathrm{I}\,\mu\nu}$ defined as usual by $\gamma_{\mathrm{I}\,\mu\nu} u^{\nu}_{\mathrm{I}} = 0$, $\gamma_{\mathrm{I}\,\mu\nu}\gamma^{\nu\rho} = \delta^{\rho}_{\mu} - t_{\mu} u^{\rho}_{\mathrm{I}}$. The formula (115) thus implies that for the constituent with label H that is pinned to the set of currents with label $\langle U\rangle$, we shall have

$$f^{\mathrm{H}}_{\{0\}\,\mu} = n_{\mathrm{H}} \left(\varpi^{\mathrm{H}}_{\mu\nu} v^{\ \nu}_{\langle U\rangle\mathrm{H}} - t_{\mu} u^{\nu}_{\mathrm{H}} \varpi^{\mathrm{H}}_{\nu\rho} u^{\ \rho}_{\langle U\rangle} \right), \tag{117}$$

with the evident implication that $u^{\ \mu}_{\langle U\rangle} f^{\mathrm{H}}_{\{0\}\,\mu}$ will vanish, and hence by the definition (72) that the second term in (117) will drop out. Thus we are finally left with the simple formula

$$f^{\mathrm{H}}_{\{0\}\,\mu} = n_{\mathrm{H}} \varpi^{\mathrm{H}}_{\mu\nu} v^{\ \nu}_{\langle U\rangle\mathrm{H}}\,, \tag{118}$$

in which the right hand side is interpretable as a gauge invariant version of the Joukowski formula for the Magnus effect. It is easy to see — using the definition (72) and the decomposition $u^{\nu}_{\langle U\rangle} = u_{\mathrm{H}}{}^{\nu} + v^{\nu}_{\langle U\rangle_{\mathrm{H}}}$ — that the application of this force law (118) is indeed equivalent to the imposition of the pinning condition (112).

It is to be remarked that the phenomena of pinning and convection are physically rather similar in that they both can be considered as constraints representing the effect of extremely strong dissipative coupling. However the way they have been dealt with here mathematically is very different. In the case of pinning the dynamical equations have been adjusted in such a way that, after having been imposed as an initial value restriction, the relevant restraint will be preserved by the equations of motion. On the other hand in the case of convection the constraint has been directly imposed at an algebraic level, so as to reduce the number of independent components of the system from $4N$, where N is the number of constituents (each with its own current 4-vector n^{μ}_{X}) to $4N - 3N'$ where N' is the number of independent comotion constraints (each of which removes the corresponding 3-velocity components from the list of independent variables while leaving the corresponding number density). The most familiar example is that of a generic non-barotropic fluid, as characterized by just a single independent velocity, so that $N' = N - 1$ and the number of independent components is just $N + 3$, including as a special case the barotropic fluid model characterized by $N = 1$, $N' = 0$, for which the number of independent components reduces to 4.

Acknowledgments

The authors wish to thank Silvano Bonazzola, Eric Gourgoulhon, David Langlois, Reinhard Prix and David Sedrakian for instructive conversations.

Appendix. The Basic Convective Superconducting Superfluid Model

In order to set up a large scale neutron star model for the purpose of describing the pulsar glitch phenomenon — which is generally recognized to depend on relative motion of a superfluid constituent relative to a normal background — the usual kind of perfect fluid model will evidently be inadequate. On the other hand it may be hoped that a satisfactory description will be attainable without recourse to the very elaborate kind of model involving separate allowance for the many degrees of freedom (such as those of the electromagnetic field that plays an essential role in the mechanics of the external magnetosphere) that would need to be taken into account in a highly accurate treatment. As a reasonable compromise, for use in such a context, the following basic convective superconducting superfluid kind of model would seem to be appropriate.

The proposed basic model has three independent constituents of which one is superfluid while the other two are subjected to a convection constraint. Thus in

the notation of the previous section it is characterized by $N = 3$ and $N' = 1$ which means that it has nine dynamically independent components.

In the context for which it is intended, the three constituent currents are to be considered as consisting of an entropy current

$$s^\mu = s u_\emptyset^\mu \,, \tag{A.1}$$

a superfluid neutron current

$$n_{\rm n}^\mu = n_{\rm n} u_{\rm n}^\mu \,, \tag{A.2}$$

and a normal current

$$n_{\rm c}^\mu = n_{\rm c} u_{\rm c}^\mu \,, \tag{A.3}$$

that convects the entropy flux, with which it shares the unit flow 4-vector

$$u_{\rm c}^\mu = u_\emptyset^\mu \,, \tag{A.4}$$

and that is to be interpreted as representing the flux of all other baryons. This normal baryon current is to be thought of as consisting not just of protons but at deeper levels also of hyperons, whose charge is neutralized by an ambient lepton gas consisting not just of electrons but at deeper levels also of muons. In the crust layers the normal baryon constituent will also include the neutrons that are confined within atomic nuclei. A realistic treatment of the crust (whose outer layers, at densities below 10^{11} g/cm^3, contain no unconfined superfluid neutrons at all) would need the use of a model of a more elaborate elastic conducting solid kind, whose formulation, in the covariant Newtonian approach developed here, will be left for future work. A Newtonian treatment will in any case be inadequate for an accurate treatment of the deeper levels, for which a relativistic version[11] of the model would of course be needed.

As in the relativistic version, there will be a baryon conservation law having the form

$$\nabla_\nu n_{\rm n}^\nu + \nabla_\nu n_{\rm c}^\nu = 0 \,. \tag{A.5}$$

In the Newtonian approximation, with which we are concerned here, this law is to be interpreted as representing the conservation of rest mass, on the understanding that both — superfluid neutron and normal (protonic or other) kinds of baryon are treated as having the same rest mass, m say, per particle, while there is of course no rest mass associated with the entropy, i.e. we have

$$m^\emptyset = 0 \,, \quad m^{\rm c} = m \,, \quad m^{\rm n} = m \,. \tag{A.6}$$

Within the few per cent level of accuracy — the most that can be expected from a Newtonian treatment in this context — this common rest mass m can be chosen indifferently to be either the mass of the hydrogen atom or simply the bare proton mass $m_{\rm p}$ (not to mention the value traditionally preferred by chemists, which is one sixteenth of the mass of an ordinary oxygen atom).

In order to characterize a particular model of this type, the essential element — which is all that is needed in the conservative case — is the pressure function Ψ or equivalently its dynamical conjugate the master function Λ_{int}, whose unconstrained version is given as a function of the 4-vectors s^μ, $n_{\text{c}}^{\ \mu}$ and $n_{\text{n}}^{\ \mu}$ in a gauge invariant manner — meaning that it can depend only on the six-scalars consisting of the number densities s, n_{c} and n_{n} together with the relative velocity magnitudes $v_{\text{c}\emptyset}$, $v_{\text{n}\emptyset}$ and v_{nc} from which the corresponding internal momentum contributions are obtainable as the partial derivatives given by the variation law

$$\delta\Lambda_{\text{int}} = \Theta_\nu \delta s^\nu + \chi^{\text{c}}_{\ \nu}\delta n_{\text{c}}^{\ \nu} + \chi^{\text{n}}_{\ \nu}\delta n_{\text{n}}^{\ \nu}\,. \tag{A.7}$$

The pressure function is obtainable from the master function, and vice versa, by a Legendre type tranformation expressible as

$$\Psi = \Lambda_{\text{int}} - \Theta_\nu s^\nu - \chi^{\text{c}}_{\ \nu} n_{\text{c}}^{\ \nu} - \chi^{\text{n}}_{\ \nu} n_{\text{n}}^{\ \nu}\,, \tag{A.8}$$

so that its variation law will have the form

$$\delta\Psi = -s^\nu \delta\Theta_\nu - n_{\text{c}}^{\ \nu}\delta\chi^{\text{c}}_{\ \nu} - n_{\text{n}}^{\ \nu}\delta\chi^{\text{n}}_{\ \nu}\,. \tag{A.9}$$

The associated stress momentum energy density tensor will be given[2] by a formula of the standard form

$$T^\mu_{\ \nu} = s^\mu\, \pi^{\emptyset}_{\ \nu} + n_{\text{c}}^{\ \mu}\, \pi^{\text{c}}_{\ \nu} + n_{\text{n}}^{\ \mu}\, \pi^{\text{n}}_{\ \nu} + \Psi\delta^\mu_{\ \nu}\,, \tag{A.10}$$

in terms of the corresponding set of complete 4-momentum covectors, of which that of the entropy is simply

$$\pi^{\emptyset}_{\ \mu} = \Theta_\mu\,, \tag{A.11}$$

while those of the massive constituents are given in terms of the corresponding frame dependent 3-velocity covectors $v_{\text{c}\,\mu} = \gamma_{\mu\nu}u_{\text{c}}^{\ \nu}$ and $v_{\text{n}\,\mu} = \gamma_{\mu\nu}u_{\text{n}}^{\ \nu}$ by

$$\pi^{\text{c}}_{\ \mu} = \chi^{\text{c}}_{\ \mu} + m v_{\text{c}\,\mu} - m\big(\tfrac{1}{2}v_{\text{c}}^{\ 2} + \phi\big)t_\mu\,, \tag{A.12}$$

$$\pi^{\text{n}}_{\ \mu} = \chi^{\text{n}}_{\ \mu} + m v_{\text{n}\,\mu} - m\big(\tfrac{1}{2}v_{\text{n}}^{\ 2} + \phi\big)t_\mu\,, \tag{A.13}$$

where ϕ is the gravitational potential.

In terms of the corresponding thermal, normal and superfluid vorticity forms, namely

$$\varpi^{\emptyset}_{\mu\nu} = 2\nabla_{[\mu}\pi^{\emptyset}_{\ \nu]}\,, \quad \varpi^{\text{c}}_{\mu\nu} = 2\nabla_{[\mu}\pi^{\text{c}}_{\ \nu]}\,, \quad \varpi^{\text{c}}_{\mu\nu} = 2\nabla_{[\mu}\pi^{\text{c}}_{\ \nu]}\,, \tag{A.14}$$

the associated 4-force covectors will be expressible by the defining formulae

$$f^{\emptyset}_{\ \mu} = s^{\,\nu}\varpi^{\emptyset}_{\nu\mu} + \Theta_\mu\nabla_\nu s^{\,\nu}\,, \tag{A.15}$$

$$f^{\text{c}}_{\ \mu} = n_{\text{c}}^{\ \nu}\varpi^{\text{c}}_{\nu\mu} + \pi^{\text{c}}_{\ \mu}\nabla_\nu n_{\text{c}}^{\ \nu}\,, \tag{A.16}$$

$$f^{\text{n}}_{\ \mu} = n_{\text{n}}^{\ \nu}\varpi^{\text{n}}_{\nu\mu} + \pi^{\text{n}}_{\ \mu}\nabla_\nu n_{\text{n}}^{\ \nu}\,. \tag{A.17}$$

It is to be remarked that if we want to describe the superfluid on a mesoscopic scale (large compared with the microscopic particle separation length scales but small compared with the intervortex spacing) we would need to impose the restraint that the superfluid vorticity $\varpi^{\rm n}_{\nu\mu}$ should vanish, but that it will in general have a nonzero value on the macroscopic scale (meaning one that is large compared with the intervortex spacing) for which the present treatment is intended.

If we are dealing with a conducting (as opposed to convective) model the information needed to characterize the dynamical evolution of the system would consist of a complete specification of all three of the 4-force covectors that have just been listed, which is equivalent to the specification of the three creation rates $\nabla_\nu s^\nu$, $\nabla_\nu n_{\rm c}^\nu$, $\nabla_\nu n_{\rm n}^\nu$, and of the three corresponding space projected 3-force vectors $\gamma^{\mu\nu} f^\emptyset_\nu$, $\gamma^{\mu\nu} f^{\rm c}_\nu$, $\gamma^{\mu\nu} f^{\rm n}_\nu$ (the simplest possibility being that of the strictly conservative case for which all three 4-force covectors are set to zero). In a convecting model of the kind we wish to consider here, we still need to specify the creation rates $\nabla_\nu s^\nu$, $\nabla_\nu n_{\rm c}^\nu$, $\nabla_\nu n_{\rm n}^\nu$, as well as the space projected 3-force vector $\gamma^{\mu\nu} f^{\rm n}_\nu$ of the superfluid constituent, but in view of the constraint (A.4) we do not need a separate specification for the corresponding thermal and normal baryon contributions, but only for their sum $\gamma^{\mu\nu} f^{\langle{\rm c}\rangle}_\nu$ where $f^{\langle{\rm c}\rangle}_\nu$ is the combined 4-force density defined by

$$f^{\langle{\rm c}\rangle}_\nu = f^{\rm c}_\nu + f^\emptyset_\nu \,. \tag{A.18}$$

The foregoing variational specification of the separate 4-momenta as functions of the corresponding currents requires that the master function be defined not just for convectively constrained configurations but even when there is a relative motion between the entropy current and the normal baryon constituent. Such a general, unconstrained specification will indeed be available if the convective (nine component) model under consideration has been obtained as a high resistivity approximation from an unconstrained (12 component) conducting in which all three constituents move independently. It will however be more economical from a mathematical point of view to avoid the introduction of redundant information from such an unconstrained ancestor model, and to work entirely within the framework of a reduced variational framework in which the master function $\Lambda_{\rm int}$ and the associated pressure function Ψ are specified only for the range of variables allowed by the convectivity constraint (A.4).

In such a reduced formulation, the nine independent components can be taken to consist of the four components of the superfluid current vector $n_{\rm n}^\nu$, the four components of the normal baryon current vector $n_{\rm c}^\nu$, together with just the density s of the entropy current, whose velocity is not independent but given by that of the normal baryon current. The most general variation that is allowed within this reduced formulation will be given by an expression of the form

$$\delta\Lambda_{\rm int} = -\Theta\,\delta s + \chi^{\langle{\rm c}\rangle}_\mu\,\delta n_{\rm c}^\nu + \chi^{\rm n}_\mu\,\delta n_{\rm n}^\nu \,, \tag{A.19}$$

which does not provide a specification of the separate thermal and normal baryon momentum covectors Θ_μ and χ_μ but only of the scalar temperature Θ and the

amalgamated normal momentum covector $\chi^{\langle \mathrm{c} \rangle}_{\mu}$ that can be evaluated within the ancestral unconstrained framework as

$$\Theta = -u^{\nu}_{\emptyset}\Theta_{\nu}\,, \qquad \chi^{\langle \mathrm{c} \rangle}_{\mu} = \chi^{\mathrm{c}} + \frac{s}{n_{\mathrm{c}}}(\Theta t_{\mu} + \Theta_{\mu})\,. \tag{A.20}$$

The corresponding complete amalgamated normal momentum covector

$$\pi^{\langle \mathrm{c} \rangle}_{\mu} = \pi^{\mathrm{c}}_{\mu} + \frac{s}{n_{\mathrm{c}}}(\Theta t_{\mu} + \Theta_{\mu}) \tag{A.21}$$

will be obtainable in the reduced formulation as

$$\pi^{\langle \mathrm{c} \rangle}_{\mu} = \chi^{\langle \mathrm{c} \rangle}_{\mu} + m v_{\mathrm{c}\,\mu} - m\big(\tfrac{1}{2}v_{\mathrm{c}}^{2} + \phi\big)t_{\mu}\,. \tag{A.22}$$

It is to be emphasized that it is possible for physically different ancestor models — as characterized by different values of Θ_{μ} and χ_{μ} for given values of the independent currents — to engender the same reduced model, in the sense of providing the same values for Θ and $\chi^{\langle \mathrm{c} \rangle}_{\mu}$, and hence also for the pressure function (A.9) and the stress energy tensor (A.22), which will be expressible as

$$\Psi = \Lambda_{\mathrm{int}} + \Theta s - \chi^{\langle \mathrm{c} \rangle}_{\nu}\, n_{\mathrm{c}}^{\nu} - \chi^{\mathrm{n}}_{\nu} n_{\mathrm{n}}^{\nu}\,, \tag{A.23}$$

and

$$T^{\mu}_{\nu} = n_{\mathrm{c}}^{\mu}\,\pi^{\langle \mathrm{c} \rangle}_{\nu} - \Theta s^{\mu} t_{\nu} + n_{\mathrm{n}}^{\mu}\,\pi^{\mathrm{n}}_{\nu} + \Psi\delta^{\mu}_{\nu}\,. \tag{A.24}$$

The formalism of the reduced formulation is not quite so elegant, but it has the advantage of avoiding the introduction of operationally redundant information singling out some particular one of the compatible unconstrained ancestor models. In the framework of the reduced formulation the combined force (A.18) will be expressible as

$$f^{\langle \mathrm{c} \rangle}_{\mu} = 2n_{\mathrm{c}}^{\nu}\nabla_{[\nu}\pi^{\langle \mathrm{c} \rangle}_{\mu]} + \pi^{\langle \mathrm{c} \rangle}_{\mu}\nabla_{\nu}n_{\mathrm{c}}^{\nu} + s\nabla_{\mu}\Theta - t_{\mu}\nabla_{\nu}(\Theta s^{\nu})\,. \tag{A.25}$$

We thereby obtain an expression of the form

$$\begin{aligned}\gamma^{\mu\nu} f^{\langle \mathrm{c} \rangle}_{\{0\}\nu} &= 2\gamma^{\mu\rho} n_{\mathrm{c}}^{\nu}\nabla_{[\nu}\pi^{\langle \mathrm{c} \rangle}{}_{\rho]} + s\gamma^{\mu\nu}\nabla_{\mu}\Theta \\ &\quad + \Big(\gamma^{\mu\nu}\big(\chi^{\langle \mathrm{c} \rangle}_{\nu} - \chi^{\mathrm{n}}{}_{\nu}\big) + m\, v^{\mu}_{\mathrm{cn}}\Big)\nabla_{\nu}n_{\mathrm{c}}^{\nu}\,,\end{aligned} \tag{A.26}$$

for the corresponding adjusted (gauge invariant) 3-force density vector, whose analogue, for the free neutron current, will be given simply by

$$\gamma^{\mu\nu} f^{\mathrm{n}}_{\{0\}\nu} = \gamma^{\mu\rho} n_{\mathrm{n}}^{\nu}\varpi^{\mathrm{n}}_{\nu\rho}\,, \quad \varpi^{\mathrm{n}}_{\nu\rho} = 2\nabla_{[\nu}\pi^{\mathrm{n}}{}_{\rho]}\,. \tag{A.27}$$

In order to complete the determination of the dynamical equations of the model, it is necessary to choose the rules specifying the values of these space projected 3-force vectors $\gamma^{\mu\nu} f^{\langle \mathrm{c} \rangle}_{\{0\}\nu}$ and $\gamma^{\mu\nu} f^{\mathrm{n}}_{\{0\}\nu}$, and of the two independent creation rates $\nabla_{\nu}s^{\nu}$ and $\nabla_{\nu}n_{\mathrm{n}}^{\nu}$ — of which the latter, by (A.5), determines $\nabla_{\nu}n_{\mathrm{c}}^{\nu}$. The simplest possibility is of course that of a strictly conservative model for which the forces and creation rates all vanish. What we want to consider here is the more general case

in which there is internal dissipation by the mechanisms described in the preceding sections and perhaps also a non-vanishing heat loss rate $\mathcal{Q}$ (due to neutrino emission) but we shall suppose that although the system may thus be open in the thermodynamic sense it is nevertheless isolated in the sense that there are no external contributions to the space projected force densities, nor to the time component of the force acting on the free neutron current, so that in the notation of the preceding section we shall have

$$f^{\rm n}_{\rm ext\,\nu} = 0\,, \qquad f^{\langle {\rm c}\rangle}_{\rm ext\,\nu} = \mathcal{Q} t_\nu\,. \tag{A.28}$$

According to the general principles developed in the preceding sections, the space projected forces will therefore be given by expressions of the standard form

$$\gamma^{\mu\nu} f^{\langle {\rm c}\rangle}_{\{0\}\nu} = f^{\langle {\rm c}\rangle {\rm n}\mu} - \nabla_\nu \tau^{\langle {\rm c}\rangle\,\mu\nu}\,, \tag{A.29}$$

$$\gamma^{\mu\nu} f^{\rm n}_{\{0\}\nu} = -f^{\langle {\rm c}\rangle {\rm n}\mu} - \nabla_\nu \tau^{{\rm n}\,\mu\nu}\,. \tag{A.30}$$

In a high temperature version of the model, the mutual interaction force density $f^{\langle {\rm c}\rangle {\rm n}\mu}$ would be given in terms of an ordinary positive resistivity coefficient $Z^{\langle {\rm c}\rangle {\rm n}}$ by an expression of the form

$$f^{\langle {\rm c}\rangle {\rm n}\mu} = Z^{\langle {\rm c}\rangle {\rm n}} v_{\rm cn}{}^{\mu}\,. \tag{A.31}$$

What we are particularly interested in here however is the low temperature version of the model, in which the free neutron current is a superfluid, which means that instead of being given by an ordinary resistivity formula of the form (A.31) the mutual interaction will be given by a vortex drag formula of the kind given by the formula (111).

In the present case, the vortex drag force density will be given by the expression

$$f^{\langle {\rm c}\rangle {\rm n}\,\mu} = -n_{\rm n}\gamma^{\mu\nu}\varpi^{\rm n}_{\nu\rho} V^{\langle {\rm c}\rangle\,\rho}_{\rm n}\,, \tag{A.32}$$

in which $c^{\langle {\rm c}\rangle}_{\rm n}$ is the positive drag coefficient, and the vector $V^{\langle {\rm c}\rangle\,\mu}_{\rm n}$ is defined by

$$V^{\langle {\rm c}\rangle\,\mu}_{\rm n} = -\frac{c^{\langle {\rm c}\rangle}_{\rm n}}{w^{\rm n}}\gamma^{\mu\nu}\varpi^{\rm n}_{\nu\rho} u^{\rho}_{\rm c}\,, \quad (w^{\rm n})^2 = \tfrac{1}{2}\gamma^{\mu\nu}\gamma^{\rho\sigma}\varpi^{\rm n}_{\mu\rho}\varpi^{\rm n}_{\nu\sigma}\,, \tag{A.33}$$

or alternatively just by

$$V^{\langle {\rm c}\rangle\,\mu}_{\rm n} = v_{\langle {\rm c}\rangle n}{}^{\mu}\,, \tag{A.34}$$

in the non-dissipative large $c^{\langle {\rm c}\rangle}_{\rm n}$ limit case of vortex pinning. In any such (pinned or unpinned) superfluid model, the free neutron current will not be subjected to any viscosity force, i.e. we shall have

$$\tau^{{\rm n}\,\mu\nu} = 0\,, \tag{A.35}$$

but for the combined thermal and normal baryon current contribution there will in general be a non-vanishing viscosity contribution of the standard form

$$\tau^{\langle {\rm c}\rangle\,\mu\nu} = -2\eta^{\langle {\rm c}\rangle}\sigma_{\rm c}{}^{\mu\nu} - \zeta^{\langle {\rm c}\rangle}\theta_{\rm c}\gamma^{\mu\nu}\,, \tag{A.36}$$

in which $\eta^{\langle \mathrm{c} \rangle}$ and $\zeta^{\langle \mathrm{c} \rangle}$ are positive shear and bulk viscosities (of which the latter will be negligibly small in many circumstances) and ${\sigma_{\mathrm{c}}}^{\mu\nu}$ and θ_{c} are the trace free and trace parts of the normal constituent's expansion tensor as given, according to (56), by

$$\theta_{\mathrm{c}}^{\mu\nu} = \gamma^{\sigma(\mu}\nabla_\sigma u_{\mathrm{c}}^{\nu)} = \sigma_{\mathrm{c}}^{\mu\nu} + \tfrac{1}{3}\theta_{\mathrm{c}}\gamma^{\mu\nu} \,. \tag{A.37}$$

To complete the specification of the model it remains to give the prescription for the creation rates. In the present case the only relevant kind of generalized chemical reaction is one whereby a free neutron is created by a process such as "dripping" out of a confined state within a neutron star crust nucleus or by inverse beta decay of a proton in the core, so that the corresponding creation numbers will be $N_{\mathrm{n}} = 1$ and $N_{\mathrm{c}} = -1$, and the corresponding chemical affinity will be given by the formula

$$\mathcal{A}_{\{\emptyset\}} = \mathcal{E}^{\mathrm{c}}_{\{\emptyset\}} - \mathcal{E}^{\mathrm{n}}_{\{\emptyset\}} \,, \tag{A.38}$$

in terms of the relevant thermal rest frame energies as defined in terms of the relevant (ancestral or reduced) 4-momentum covectors by

$$\mathcal{E}^{\mathrm{c}}_{\{\emptyset\}} = -u^\nu_\emptyset \pi^{\mathrm{c}}{}_\nu = -u^\nu_\emptyset \pi^{\langle \mathrm{c} \rangle}_\nu \,, \quad \mathcal{E}^{\mathrm{n}}_{\{\emptyset\}} = -u^\nu_\emptyset \pi^{\mathrm{n}}{}_\nu \,. \tag{A.39}$$

The superfluid particle creation rate can thus be seen to be given in terms of the relevant transfusion coefficient κ by an expression of the standard form

$$\nabla_\nu n^\nu_{\mathrm{n}} = \kappa \, \mathcal{A}_{\{\emptyset\}} \,, \tag{A.40}$$

in which the affinity is given in terms of the relative flow velocity magnitude v_{nc} by the manifestly frame independent formula

$$\mathcal{A}_{\{\emptyset\}} = \chi^{\mathrm{c}}_\natural - \chi^{\mathrm{n}}_\emptyset - \tfrac{1}{2} m v^2_{\mathrm{nc}} \,, \tag{A.41}$$

where

$$\chi^{\mathrm{c}}_\natural = -u_{\mathrm{c}}^\nu \chi^{\mathrm{c}}_\nu = -u_{\mathrm{c}}^\nu \chi^{\langle \mathrm{c} \rangle}_\nu \,, \quad \chi^{\mathrm{n}}_\emptyset = -u^\nu_\emptyset \chi^{\mathrm{n}}_\nu = -u_{\mathrm{c}}^\nu \chi^{\mathrm{n}}_\nu \,. \tag{A.42}$$

The last thing we need to complete the specification of the model is the value of the energy emission rate $\mathcal{Q}$ that determines the entropy rate via the formula (97), which gives

$$\begin{aligned} \mathcal{Q} + \Theta \nabla_\mu s^\mu = {} & \kappa \mathcal{A}^2_{\{\emptyset\}} + 2\eta^{\langle \mathrm{c} \rangle} \gamma_{\mu\nu}\gamma_{\rho\sigma}\sigma_{\mathrm{c}}^{\mu\rho}\sigma_{\mathrm{c}}^{\nu\sigma} \\ & + \zeta^{\langle \mathrm{c} \rangle}\theta^2_{\mathrm{c}} + \frac{n_{\mathrm{n}} w^{\mathrm{n}}}{c^{\langle \mathrm{c} \rangle}_{\mathrm{n}}} V^{\langle \mathrm{c} \rangle \, \mu}_{\mathrm{n}} \gamma_{\mu\nu} V^{\langle \mathrm{c} \rangle \, \nu}_{\mathrm{n}} \,. \end{aligned} \tag{A.43}$$

The complete set of dynamical equations for the nine independent components (those of the space vectors v_{c}^μ and v_{n}^μ together with the scalars n_{c}, n_{n} and s) is thus completed: it consists of the creation formulae (A.5), (A.40) and (A.43), together with the pair of 3-force equations (A.29), (A.30) (as made explicit by the prescriptions (A.32), (A.33), (A.35) and (A.36) for drag and viscosity).

It is to be remarked that the final term in (A.43) will drop out in the large $c_{\rm n}^{\langle{\rm c}\rangle}$ limit for which the drag prescription (A.33) is replaced by the vortex pinning prescription (A.33). In a similar way the first term in (A.43) in the large κ limit for which the system will be maintained in a state of chemical equilibrium as characterized by the condition $\mathcal{A}_{\{\emptyset\}} = 0$ which will have the effect of reducing the number of dynamically independent components from nine to eight.

The particular case of a thermodynamically closed model — which may be a good approximation for processes occurring on a short timescale — will be obtained by setting $\mathcal{Q} = 0$. However it will often be more realistic to take $\mathcal{Q}$ to have a value that is positive and monotonically increasing as a function of the temperature Θ to allow for losses by URCA type neutrino emission processes. For processes ocurring over a sufficiently long timescale the temperature sensitivity of $\mathcal{Q}$ near some emission threshold value may be sufficient to justify the use of a simplifying approximation whereby the temperature Θ is held fixed at the threshold value in question, thereby determining the value of s and hence of the creation rate $\nabla_\mu s^\mu$. This will reduce the number of dynamically independent components from nine to eight (or, in the chemical equilibrium case, from eight to seven) so that (A.43) will no longer be needed as a dynamical equation of the system, but will merely serve for the purpose of calculating the corresponding value of $\mathcal{Q}$ in case it might be needed. A simple extreme special case of such a fixed temperature thermodynamically open variant of the model is the zero temperature limit for which Θ and s both vanish.

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