Entrainment parameters in a cold superfluid neutron star core

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Hydrodynamic simulations of neutron star cores that are based on a two-fluid description in terms of a neutron-proton superfluid mixture require the knowledge of the Andreev-Bashkin entrainment matrix which relates the momentum of one constituent to the currents of both constituents. This matrix is derived for arbitrary nuclear asymmetry at zero temperature and in the limits of small relative currents in the framework of the energy density functional theory. The Skyrme energy density functional is considered as a particular case. General analytic formulas for the entrainment parameters and various corresponding effective masses are obtained. These formulas are applied to the liquid core of a neutron star composed of homogeneous plasma of nucleons, electrons, and possibly muons in β equilibrium.

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I. INTRODUCTION

In a standard model of a neutron star core, matter is a uniform plasma consisting of neutrons of number density n_n, and a small admixture of protons and electrons of number densities n_p and n_e, respectively. Electrons ensure the overall stability of the star by the condition of electroneutrality, $n_e = n_p$, but play a negligible role in mass transport because their mass is very small compared to the nucleon mass. If the electron Fermi energy exceeds the muon rest mass, muons are present in matter, but their density is always smaller than that of electrons. The electroneutrality condition is then $n_e + n_\mu = n_p$. The electrically charged particles are strongly coupled to the magnetically braked solid crust. This builds up a lag between the neutrons and the protons which is only restored through glitch events. Neutron star cores are therefore described within two-fluid models in terms of neutron and proton components which are superfluid in some density range when the temperature of the star falls below the corresponding critical temperatures.

As a result of the nucleon-nucleon interactions, the momentum $\boldsymbol{\pi}_q$ of each nucleon is not simply given by the corresponding velocity $\boldsymbol{v}_q$ (we use the convention that $q = n, p$ for neutron, proton, respectively) times the mass m (in the following we shall neglect the small mass difference between neutrons and protons), but in general it is a linear combination of the neutron and proton velocities. This is the so-called Andreev-Bashkin entrainment effect [1]. The discussions in the literature have been usually obscured by the confusion between momentum and velocity. Traditionally, one introduces "superfluid velocities" $\boldsymbol{V}_q$ defined by

$$\boldsymbol{V}_q = \pi_q/m, \tag{1}$$

in which it is recalled that the momenta π_q are defined by the partial derivative with respect to the nucleon current $n_q v_q$ of the Lagrangian density $\Lambda(n_q, n_q \boldsymbol{v}_q)$ of the system [2,3]. The mass current of some given nucleon species q

$$\boldsymbol{\rho}_q = \rho_q \boldsymbol{v}_q, \qquad \rho_q = n_q m \tag{2}$$

is then expressible as

$$\boldsymbol{\rho}_q = \sum_{q'} \rho_{qq'} \boldsymbol{V}_{q'}, \tag{3}$$

in which $\rho_{qq'}$ is the (symmetric) entrainment or mass density matrix. Only one of these matrix elements has to be specified since the other elements can be obtained from the identities due to Galilean invariance

$$\rho_{nn} + \rho_{np} = \rho_n, \qquad \rho_{pp} + \rho_{pn} = \rho_p. \tag{4}$$

This matrix is a necessary ingredient in dynamic simulations of neutron star cores, such as, for instance, the study of oscillation modes. The (static) equation of state and the entrainment matrix are usually obtained using different microscopic models. In earlier calculations and even recently, the mass density matrix was postulated to have some density dependence whose parameters are determined from rough estimates.

Comer *et al.* [4] have built a self-consistent equation of state in the framework of a minimal relativistic σ-ω mean field model, ignoring nonlinear couplings between the meson fields; these couplings, however, are essential in order to reproduce nuclear properties such as the incompressibility of nuclear matter. They obtained semianalytical formulas for the entrainment parameters in the limit of small fluid velocities (compared to that of light), which even within this simple mean field model take a rather complicated form. It is not clear that analytical formulas could still be obtained with realistic relativistic mean field models, taking into account self meson couplings and including as well the ρ meson which is required for a correct treatment of the symmetry energy.

Despite the fact that nonrelativistic mean field models have been widely applied in the study of terrestrial nuclei and neutron stars, there has been no attempt so far to apply these models to the calculation of the mass density matrix. The purpose of the present work is therefore to fill this gap and to further investigate the density dependence of the various entrainment parameters and effective masses that have been introduced in the literature.

II. ENTRAINMENT IN A MIXTURE OF FERMI LIQUIDS

At zero temperature, entrainment effects have been shown to be independent of the nucleon pairing correlations giving rise to superfluidity [5,6]. Even at finite temperatures well below the critical temperatures for the onset of superfluidity, pairing as well as thermal effects are very small [6]. We can therefore ignore pairing interactions and restrict ourself to the limit of zero temperature.

Borumand *et al.* [7] have shown how to obtain the entrainment matrix of a neutron-proton mixture in the framework of the Landau Fermi liquid theory. In what follows, we will limit ourselves to spin-unpolarized nuclear matter. Therefore, spin indices will not appear in our formulas, and all quantities are to be understood as spin averages. Under our assumptions, the change in the total energy density of the system due to a small current is expressed as

$$\mathcal{E} = 2\sum_q \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{(q)}(\boldsymbol{k})\delta\tilde{n}^{(q)}(\boldsymbol{k}) + 2\sum_{q,q'} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \times \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} f^{qq'}(\boldsymbol{k},\boldsymbol{k}')\delta\tilde{n}^{(q)}(\boldsymbol{k})\delta\tilde{n}^{(q')}(\boldsymbol{k}'), \tag{5}$$

in which $e^{(q)}(\mathbf{k})$ is the energy of a quasiparticle ($q = n, p$ for neutron and proton, respectively) of wave vector $\boldsymbol{k}$, and $f^{qq'}(\boldsymbol{k},\boldsymbol{k}')$ is the (spin-averaged) interaction between the quasiparticles. Moreover, $\delta\tilde{n}^{(q')}(\boldsymbol{k}')$ denotes the change in the distribution function of quasiparticle states from that of the static (zero current) ground state characterized by the Heaviside functions $\Theta(k_F^{(q)} - k)$, where $k_F^{(q)}$ is the Fermi momentum (in units of $\hbar$) $k_F^{(q)} = (3\pi^2 n_q)^{1/3}$. In the presence of neutron and proton currents, the corresponding Fermi surfaces are displaced by a vector $\boldsymbol{Q}_q$. In the limit of small currents $Q_q \ll k_F^{(q)}$ and writing the superfluid velocities from (1) as

$$\boldsymbol{V}_q = \hbar\boldsymbol{Q}_q/m, \tag{6}$$

it can be shown that the mass current $\boldsymbol{\rho}_q = \rho_q v_q$ of each nucleon species is linearly related to both the neutron and proton superfluid velocities

$$\rho_q = \sum_{q'} \rho_{qq'}\boldsymbol{V}_{q'}, \tag{7}$$

where the (symmetric) entrainment matrix $\rho_{qq'}$ is given by

$$\rho_{qq'} = \sqrt{\rho_q\rho_{q'}}\frac{m}{\sqrt{m_q^\oplus m_{q'}^\oplus}}\left(\delta_{qq'} + \mathcal{F}_1^{qq'}/3\right). \tag{8}$$

The (Landau) effective mass $m_q^\oplus$ and the dimensionless Landau parameters $\mathcal{F}_\ell^{qq'}$ are defined, respectively, by

$$\frac{1}{m_q^\oplus} = \frac{1}{\hbar^2 k_F^{(q)}}\left.\frac{de}{dk}\right|_{k=k_F^{(q)}}, \tag{9}$$

$$\mathcal{F}_\ell^{qq'} = \sqrt{\mathcal{N}_q\mathcal{N}_{q'}}f_\ell^{qq'}, \tag{10}$$

in which $\mathcal{N}_q$ is the density of quasiparticle states at the Fermi surface,

$$\mathcal{N}_q = \frac{m_q^\oplus k_F^{(q)}}{\hbar^2\pi^2}, \tag{11}$$

and the parameters $f_\ell^{qq'}$ are obtained from the Legendre expansion of the spin-averaged quasiparticle interaction,

$$f^{qq'}(\boldsymbol{k},\boldsymbol{k}') = \sum_\ell f_\ell^{qq'}P_\ell(\cos\theta), \tag{12}$$

where θ is the angle between the wave vectors $\boldsymbol{k}$ and $\boldsymbol{k}'$ lying on the corresponding Fermi surface.

Alternative formulas for the entrainment matrix [8] have been used in the literature, based on the decomposition of the Landau effective masses in the form

$$m_n^\oplus = m + \delta m_{nn}^\oplus + \delta m_{np}^\oplus, \tag{13}$$

$$m_p^\oplus = m + \delta m_{pp}^\oplus + \delta m_{pn}^\oplus, \tag{14}$$

where the various contributions to the effective masses are related to the Landau parameters by the simple formula [9]

$$\delta m_{qq'}^\oplus = \frac{1}{3}\mathcal{F}_1^{qq'}m\sqrt{\frac{n_{q'}m_q^\oplus}{n_q m_{q'}^\oplus}}. \tag{15}$$

The mass density matrix can then be equivalently written explicitly as

$$\rho_{nn} = \rho_n\frac{m + \delta m_{nn}^\oplus}{m_n^\oplus}, \tag{16}$$

$$\rho_{pp} = \rho_p\frac{m + \delta m_{pp}^\oplus}{m_p^\oplus}, \tag{17}$$

$$\rho_{np} = \rho_{pn} = \rho_n\frac{\delta m_{np}^\oplus}{m_n^\oplus} = \rho_p\frac{\delta m_{pn}^\oplus}{m_p^\oplus}. \tag{18}$$

It should be remarked that in the formulas provided by Sauls (see [8]), the terms proportional to $\delta m_{nn}^\oplus$ and $\delta m_{pp}^\oplus$ in the expressions for ρ_{nn} and ρ_{pp} are omitted, and therefore those formulas violate Galilean invariance.

The quasiparticle energies $e^{(q)}(\boldsymbol{k})$ and the quasiparticle interaction $f^{qq'}(\boldsymbol{k},\boldsymbol{k}')$ can be deduced from a microscopic approach. The solution of the many-body problem, starting from the bare nucleon-nucleon interactions, is very difficult. We shall here adopt a simpler approach based on self-consistent mean field models with phenomenological effective interactions (for a review, see for instance [10]), which have been very successful in describing the nuclear properties of terrestrial nuclei. Such mean field models have also been widely applied in the context of neutron stars.

III. LANDAU PARAMETERS IN THE ENERGY DENSITY FUNCTIONAL THEORY

We shall calculate in this section the Landau parameters for asymmetric nuclear matter in the framework of the Hohenberg-Kohn-Sham energy density functional theory [11,12].

The energy density functional for spin-unpolarized homogeneous nuclear matter is written as a sum of the isoscalar ($T=0$) and isovector ($T=1$) terms [10]

$$\mathcal{E} = \sum_{T=0,1} \delta_{T0} \frac{\hbar^2}{2m}\tau_T + C^n_T(n_b) n_T^2 + C^\tau_T n_T \tau_T + C^j_T \boldsymbol{j}_T{}^2. \quad (19)$$

The isoscalar and isovector parts of some quantity for a nucleon system are given, respectively, by the sum and the difference between the neutron and proton contributions. For example, the isoscalar and isovector densities are given by $n_0 = n_n + n_p = n_b$ and $n_1 = n_n - n_p$, respectively.

The nucleon density n_q, kinetic energy density τ_q (in units of $\hbar^2/2m$), and nucleon current $\boldsymbol{j}_q$ are expressible in terms of the nucleon distribution function $\tilde{n}^{(q)}\{\boldsymbol{k}\}$ by

$$n_q = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \tilde{n}^{(q)}(\boldsymbol{k}), \quad (20)$$

$$\tau_q = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} k^2 \tilde{n}^{(q)}(\boldsymbol{k}), \quad (21)$$

$$j_q = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \boldsymbol{k} \tilde{n}^{(q)}(\boldsymbol{k}). \quad (22)$$

Energy density functionals of the form (19) can be obtained in the Hartree-Fock approximation with effective contact nucleon-nucleon interactions $\hat{v}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ of the Skyrme type, whose standard parametrizations (ignoring spin-orbit terms which are irrelevant in the present case) are

$$\begin{aligned}\hat{v}(\boldsymbol{r}_1, \boldsymbol{r}_2) &= t_0(1 + x_0 \hat{P}_\sigma)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \\ &\quad + \frac{1}{2} t_1 (1 + x_1 \hat{P}_\sigma)(\hat{\boldsymbol{k}}^{\dagger 2} \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\hat{\boldsymbol{k}}^2) \\ &\quad + t_2(1 + x_2 \hat{P}_\sigma)\hat{\boldsymbol{k}}^\dagger \cdot \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\hat{\boldsymbol{k}} \\ &\quad + \frac{1}{6} t_3(1 + x_3 \hat{P}_\sigma)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) n_b \left(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}\right)^\gamma, \end{aligned} \quad (23)$$

where $\hat{P}_\sigma = (1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)/2$ is the spin exchange operator and $\hat{\boldsymbol{k}} = -i(\nabla_1 - \nabla_2)/2$. The density-dependent term proportional to t_3 represents the effects of three-body interactions. The coefficients $C^n_T(n_b)$, C^τ_T, and C^j_T can then be expressed in terms of the parameters of the Skyrme interaction (see appendix). It should be stressed, however, that the functional (19) is more general than the Skyrme functional. In particular, the coefficients $C^n_T(n_b)$ can be any function of the baryon density n_b.

The single-particle energies are obtained from the functional derivative of the energy density

$$e^{(q)}(\boldsymbol{k}) = \left.\frac{\delta \mathcal{E}}{\delta \tilde{n}^{(q)}(\boldsymbol{k})}\right|_0, \quad (24)$$

where the zero subscript indicates that the functional derivative is evaluated in the static ground state (in which the currents $\boldsymbol{j}_q$ vanish), characterized by the distribution function

$$\tilde{n}_0^{(q)}(\boldsymbol{k}) = \Theta\big(\boldsymbol{k}_F^{(q)} - \boldsymbol{k}\big). \quad (25)$$

Substituting the functional (19) yields

$$e_q(k) = \frac{\hbar^2 k^2}{2m_q^\oplus} + U_q, \quad (26)$$

in which the effective mass $m_q^\oplus$ [using the same symbol as for the Landau effective mass defined by (9) since both definitions coincide] and the single-quasiparticle potential U_q are given by

$$\frac{\hbar^2}{2m_q^\oplus} = \frac{\delta\mathcal{E}}{\delta\tau_q} = \frac{\hbar^2}{2m} + \big(C_0^\tau - C_1^\tau\big)n_b + 2C_1^\tau n_q, \quad (27)$$

$$\begin{aligned} U_q = \frac{\delta\mathcal{E}}{\delta n_q} &= 4C_1^n n_q - 2n_b\big(C_0^n - C_1^n\big) + \big(C_0^\tau - C_1^\tau\big)\tau_b \\ &\quad + 2C_1^\tau \tau_q + \frac{dC_0^n}{dn_b} n_b^2 + \frac{dC_1^n}{dn_b}(2n_q - n_b)^2, \end{aligned} \quad (28)$$

where $\tau_b = \tau_n + \tau_p$.

The quasiparticle interaction, calculated as a second functional derivative of the energy functional (19), is

$$f^{qq'}(\boldsymbol{k}, \boldsymbol{k}') = \left.\frac{\delta^2\mathcal{E}}{\delta\tilde{n}^{(q)}(\boldsymbol{k})\delta\tilde{n}^{(q')}(\boldsymbol{k}')}\right|_0, \quad (29)$$

and contains only $\ell = 0$ and $\ell = 1$ components.

The nonvanishing Landau parameters are found to be expressible as

$$\begin{aligned} f_0^{nn} &= 2k_F^{(n)2}\big(C_0^\tau + C_1^\tau\big) + 2\big(C_1^n + C_0^n\big) \\ &\quad + \frac{dC_0^n}{dn_b}4n_b + 4\frac{dC_1^n}{dn_b}(n_n - n_p) \\ &\quad + \frac{d^2C_0^n}{dn_b^2}n_b^2 + \frac{d^2C_1^n}{dn_b^2}(n_n - n_p)^2, \end{aligned} \quad (30)$$

$$\begin{aligned} f_0^{pp} &= 2k_F^{(p)2}\big(C_0^\tau + C_1^\tau\big) + 2\big(C_1^n + C_0^n\big) \\ &\quad + \frac{dC_0^n}{dn_b}4n_b - 4\frac{dC_1^n}{dn_b}(n_n - n_p) \\ &\quad + \frac{d^2C_0^n}{dn_b^2}n_b^2 + \frac{d^2C_1^n}{dn_b^2}(n_n - n_p)^2, \end{aligned} \quad (31)$$

$$\begin{aligned} f_0^{np} = f_0^{pn} &= \big(k_F^{(n)2} + k_F^{(p)2}\big)\big(C_0^\tau - C_1^\tau\big) + 2\big(C_0^n - C_1^n\big) \\ &\quad + \frac{dC_0^n}{dn_b}4n_b + \frac{d^2C_0^n}{dn_b^2}n_b^2 + \frac{d^2C_1^n}{dn_b^2}(n_n - n_p)^2, \end{aligned} \quad (32)$$

$$f_1^{nn} = 2\big(C_0^j + C_1^j\big)k_F^{(n)2}, \quad (33)$$

$$f_1^{pp} = 2\big(C_0^j + C_1^j\big)k_F^{(p)2}, \quad (34)$$

$$f_1^{np} = f_1^{pn} = 2\big(C_0^j - C_1^j\big)k_F^{(n)}k_F^{(p)}. \quad (35)$$

These formulas agree with those of Bender *et al.* [13] for the limiting case of symmetric nuclear matter (using standard notations $f_\ell^{nn} = f_\ell^{pp} = f_\ell + f_\ell'$ and $f_\ell^{np} = f_\ell - f_\ell'$) and generalize the results of Blaizot and Haensel [14] for asymmetric nuclear matter to any energy density functional of the form (19).

It should be remarked in particular that for any such functional (19), the parameters f_1^{nn} and f_1^{pp} are related by

$$\frac{f_1^{nn}}{f_1^{pp}} = \left(\frac{n_n}{n_p}\right)^{2/3}. \tag{36}$$

The corresponding dimensionless $\ell = 1$ Landau parameters can be expressed in compact form as

$$\mathcal{F}_1^{qq'} = 3\tilde{\alpha}_{qq'}\sqrt{n_q m_q^{\oplus} n_{q'} m_{q'}^{\oplus}}, \tag{37}$$

in which the coefficients $\widetilde{\alpha}_{qq'}$ are defined by

$$\widetilde{\alpha}_{nn} = \widetilde{\alpha}_{pp} = \frac{2}{\hbar^2}\left(C_0^j + C_1^j\right), \tag{38}$$

$$\widetilde{\alpha}_{np} = \widetilde{\alpha}_{pn} = \frac{2}{\hbar^2}\left(C_0^j - C_1^j\right). \tag{39}$$

We conclude this section by remarking that in a general case of asymmetric nuclear matter (i.e., with $n_n \neq n_p$) the $\ell = 1$ Landau parameters can be uniquely determined in terms solely of the effective masses as

$$\mathcal{F}_1^{np} = \frac{3}{m}\frac{\sqrt{n_n m_n^{\oplus} n_p m_p^{\oplus}}}{n_p^2 - n_n^2}\left[n_p\left(1 - \frac{m}{m_n^{\oplus}}\right) - n_n\left(1 - \frac{m}{m_p^{\oplus}}\right)\right], \tag{40}$$

$$\mathcal{F}_1^{nn} = 3\frac{n_n}{n_p^2 - n_n^2}\frac{m_n^{\oplus}}{m}\left[n_p\left(1 - \frac{m}{m_p^{\oplus}}\right) - n_n\left(1 - \frac{m}{m_n^{\oplus}}\right)\right], \tag{41}$$

$$\mathcal{F}_1^{pp} = 3\frac{n_p}{n_p^2 - n_n^2}\frac{m_p^{\oplus}}{m}\left[n_p\left(1 - \frac{m}{m_p^{\oplus}}\right) - n_n\left(1 - \frac{m}{m_n^{\oplus}}\right)\right]. \tag{42}$$

IV. ENTRAINMENT MATRIX AND EFFECTIVE MASSES

Subtituting the expressions (37) of the Landau parameters obtained in the previous section, the entrainment matrix elements (8) can be seen to be expressible as

$$\rho_{qq'} = \rho_q\frac{m}{m_q^{\oplus}}\delta_{qq'} + \tilde{\alpha}_{qq'}\rho_q\rho_{q'}, \tag{43}$$

or more explicitly in terms of the mass densities ρ_n and ρ_p,

$$\rho_{nn} = \rho_n(1 - \tilde{\alpha}_{np}\rho_p), \tag{44}$$

$$\rho_{pp} = \rho_p(1 - \tilde{\alpha}_{np}\rho_n), \tag{45}$$

$$\rho_{np} = \tilde{\alpha}_{np}\rho_n\rho_p. \tag{46}$$

It is readily seen that the formulas (44)–(46) imply a basic property of the entrainment matrix, namely, $\rho_{nn} + \rho_{np} = \rho_n$ and $\rho_{pp} + \rho_{np} = \rho_p$, which guarantees the Galilean invariance of the two-fluid model.

Two other kinds of effective masses, different from the Landau effective masses $m_q^{\oplus}$, have been introduced in the literature. Effective nucleon masses can be defined from the mass density matrix elements by setting

$$\frac{\rho_{qq}}{\rho_q} = \frac{m}{m_\sharp^q} \tag{47}$$

in such a way that in the proton *momentum* rest frame ($\boldsymbol{V}_p = 0$) we have $\boldsymbol{\pi}_n = m_\sharp^n \boldsymbol{v}_n$ and similarly in the neutron *momentum* rest frame, $\boldsymbol{\pi}_p = m_\sharp^p \boldsymbol{v}_p$.

These effective masses have a very simple density dependence, as shown in the formulas

$$\frac{m_\sharp^n}{m} = \frac{1}{1 - \tilde{\alpha}_{np}\rho_p}, \tag{48}$$

$$\frac{m_\sharp^p}{m} = \frac{1}{1 - \tilde{\alpha}_{np}\rho_n}. \tag{49}$$

The $\sharp$-effective masses differ from the Landau quasiparticle effective masses and are related to the latter ones by

$$\frac{m}{m_\sharp^q} = \frac{m}{m_q^{\oplus}} + \tilde{\alpha}_{qq}\rho_q, \tag{50}$$

due to the nonvanishing quasiparticle interactions. The extra term on the right hand side can be interpreted as resulting from the backflow induced by the motion of the quasiparticles.

Alternatively, one can introduce effective masses $m_\star^q$ such that in the proton rest frame (meaning $\boldsymbol{v}_p = 0$) we have $\boldsymbol{\pi}_n = m_\star^n \boldsymbol{v}_n$ and similarly in the neutron rest frame $\boldsymbol{\pi}_p = m_\star^p \boldsymbol{v}_p$. These effective masses are given by

$$\frac{m_\star^n}{m} = \frac{1 - \tilde{\alpha}_{np}\rho_n}{1 - \tilde{\alpha}_{np}\rho_b}, \tag{51}$$

$$\frac{m_\star^p}{m} = \frac{1 - \tilde{\alpha}_{np}\rho_p}{1 - \tilde{\alpha}_{np}\rho_b}. \tag{52}$$

where $\rho_b = \rho_n + \rho_p$. The effective masses of the different kinds are related by

$$m_\sharp^n - m = (m_\star^n - m)\left[1 + \frac{n_n}{n_p}\left(\frac{m_\star^n}{m} - 1\right)\right]^{-1}, \tag{53}$$

$$m_\sharp^p - m = (m_\star^p - m)\left[1 + \frac{n_p}{n_n}\left(\frac{m_\star^p}{m} - 1\right)\right]^{-1}. \tag{54}$$

These formulas show that, as pointed out by Prix *et al.* [15], in the limit of a very small proton fraction $n_p/n_b \ll 1$, as relevant in the liquid core of neutron stars, we will have $m_\star^n \sim m_\sharp^n \sim m$ and $m_\star^p \sim m_\sharp^p$. We will compute more accurate values of the effective masses in Sec. VI.

In studies of neutron star cores, the nondiagonal entrainment matrix element ρ_{np} has often been parametrized as

$$\rho_{np} = -\epsilon\rho_n, \tag{55}$$

in which the dimensionless parameter ϵ was taken as a constant [16–18]. Other authors [15,19,20] have suggested instead to set the dimensionless parameters defined by

$$\varepsilon_q = 1 - \frac{m_\star^q}{m}, \tag{56}$$

as constants. However, comparison with (46), (51), and (52) shows that neither ansatz is satysfying, since these parameters are found to vary with the densities according to

$$\epsilon = -\tilde{\alpha}_{np}\rho_p, \tag{57}$$

$$\varepsilon_n = -\frac{\tilde{\alpha}_{np}\rho_p}{1-\tilde{\alpha}_{np}\rho_b}, \tag{58}$$

$$\varepsilon_p = -\frac{\tilde{\alpha}_{np}\rho_n}{1-\tilde{\alpha}_{np}\rho_b}. \tag{59}$$

The effective masses and entrainment parameters seem to diverge at some points of the ρ_n-ρ_p plane. However, as will be shown in the next section, once stability constraints are imposed, these apparent singularities disappear.

V. STABILITY OF THE STATIC GROUND STATE AND CONSTRAINTS ON THE ENTRAINMENT PARAMETERS

Since the momentum of each nucleon is a linear combination of both the neutron and proton currents, this means that the corresponding dynamical contribution to the Lagrangian density of the system $\Lambda_{\rm dyn} = \mathcal{E}_{\rm dyn}$ is a bilinear symmetric form of the currents. It is therefore readily seen that this dynamical contribution is expressible in terms of the Andreev-Bashkin entrainment matrix elements as

$$\mathcal{E}_{\rm dyn} = \tfrac{1}{2}\big(\rho_{nn}\boldsymbol{V}_n^2 + 2\rho_{np}\boldsymbol{V}_n\cdot\boldsymbol{V}_p + \rho_{pp}\boldsymbol{V}_p^2\big). \tag{60}$$

As a result, the total energy density $\mathcal{E}$ of the fluid mixture can be written as the sum of the dynamical contribution $\mathcal{E}_{\rm dyn}$ and an internal static contribution $\mathcal{E}_{\rm ins}$ which only depends on the densities: $\mathcal{E} = \mathcal{E}_{\rm dyn} + \mathcal{E}_{\rm ins}$.

The static ground state of the system is stable if the term $\mathcal{E}_{\rm dyn}$ is strictly positive. This means that the entrainment matrix must be positive definite (the minimum energy state thus being obtained by the vanishing of the superfluid velocities or equivalently of the currents, i.e., $\mathcal{E}_{\rm dyn} = 0$), which means that its eigenvalues must be strictly positive. This condition entails that the matrix elements (44)–(46) should obey

$$\rho_{nn} + \rho_{pp} > 0, \tag{61}$$

$$\rho_{np}^2 < \rho_{nn}\rho_{pp}. \tag{62}$$

These conditions lead to contraints on the $\ell = 1$ Landau parameters (using the other constraint that the Landau effective masses $m_n^{\oplus}$ and $m_p^{\oplus}$ have to be positive),

$$\mathcal{F}_1^{nn} > -3, \qquad \mathcal{F}_1^{pp} > -3, \tag{63}$$

together with

$$\left(1+\frac{1}{3}\mathcal{F}_1^{nn}\right)\left(1+\frac{1}{3}\mathcal{F}_1^{pp}\right) > \left(\frac{\mathcal{F}_1^{np}}{3}\right)^2. \tag{64}$$

The stability conditions can also be expressed in terms of effective masses [21]

$$\frac{m_\star^q}{m} > \frac{n_q}{n_b}, \tag{65}$$

or equivalently

$$\frac{m_\sharp^q}{m} < \frac{n_b}{n_q}. \tag{66}$$

In terms of the dimensionless entrainment parameters ε_q, these conditions can be expressed as

$$\varepsilon_q < 1 - \frac{n_q}{n_b}. \tag{67}$$

It should be remarked that the previous inequalities are very general and have to be satisfied in *any* two-fluid model. In the present case, these conditions also impose a constraint on the energy functional (19) from which the entrainment matrix is derived. Since the conditions (61) and (62) must be satisfied for any neutron and proton densities, this leads to the following requirement

$$C_0^j \leqslant C_1^j. \tag{68}$$

In the particular case of Skyrme functionals, this last condition reads

$$t_1(2+x_1) + t_2(2+x_2) \geqslant 0. \tag{69}$$

Whenever this condition is fulfilled, it can be seen from Eqs. (48)–(52) that for any neutron and proton densities, the effective masses $m_\star^q$ and $m_\sharp^q$ are therefore positive and smaller than the bare nucleon mass; that is,

$$0 < m_\star^q, m_\sharp^q \leqslant m. \tag{70}$$

Combining the latter inequality with (65) shows in particular that

$$n_q/n_b < m_\star^q/m \leqslant 1. \tag{71}$$

Besides, since the proton fraction is very small inside neutron stars, it can be seen from Eq. (68) and the definitions (48), (49), (51), and (52) that in this case the neutron effective masses are always larger than the proton ones,

$$m_\star^n > m_\star^p, \qquad m_\sharp^n > m_\sharp^p. \tag{72}$$

Likewise, it can be shown that $\varepsilon_q \geqslant 0$ which, in association with (67), yields

$$0 \leqslant \varepsilon_q < 1 - \frac{n_q}{n_b}. \tag{73}$$

It is thus found that the entrainment parameters are well-behaving functions of nucleon densities.

VI. APPLICATION TO NEUTRON STAR MATTER IN β EQUILIBRIUM

In the previous section, we obtained general formulas for the entrainment parameters and associated effective masses for nuclear matter with arbitrary asymmetry. In the present section, we will apply these formulas to construct a model of a neutron star core. We will use the SLy4 Skyrme force specifically devised for astrophysical purposes [22–25]. In the framework of a compressible liquid drop model based on the SLy4 Skyrme energy functional, Douchin and Haensel [26] found that the bottom edge of the crust corresponds to the baryon density $n_{\rm edge} \simeq 0.076\ \rm fm^{-3}$. In the following, we will consider the density domain $n_{\rm edge} < n_b < 3n_{\rm s}$, where $n_{\rm s} = 0.16\ \rm fm^{-3}$ is the nuclear saturation density.

We assume that the liquid core is composed of a homogeneous plasma of neutrons, protons, and electrons (and muons for baryon densities beyond some critical threshold) in β equilibrium,

$$n \leftrightarrow p^+ + e^- + \nu_e, \qquad \mu^- \leftrightarrow e^- + \nu_\mu + \bar{\nu}_e. \tag{74}$$

This means that the chemical potentials of the various species are related by (assuming that neutrinos escaped from the star)

$$\mu_n = \mu_p + \mu_e, \qquad \mu_e = \mu_\mu. \tag{75}$$

In the Hartree-Fock approximation, chemical potentials of nucleons are equal to the corresponding Fermi energies ($q = n, p$) including the rest mass energy

$$\mu_q = m_q c^2 + \frac{\hbar^2 k_F^{(q)2}}{2m_q^\oplus} + U_q. \tag{76}$$

Considering the leptons as ideal relativistic Fermi gases, the lepton chemical potentials are given by ($l = e, \mu$)

$$\mu_l = \sqrt{m_l c^2 + \hbar^2 c^2 (3\pi^2 n_l)^{2/3}}. \tag{77}$$

Charge neutrality requires that

$$n_p = n_e + n_\mu. \tag{78}$$

In Eqs. (76) and (77), we neglected the deviations in the chemical potentials caused by the existence of nonvanishing currents, since the relative velocities are typically very small compared to the velocities of the various constituents. For completeness, let us mention that the internal energy density of the nucleons can be decomposed in the form

$$\mathcal{E}_{\rm int} = \mathcal{E}_0 + \mathcal{E}_{\rm ent}, \tag{79}$$

in which $\mathcal{E}_0$ is the functional (19) evaluated in the static ground state with the distribution function (25) including the rest mass

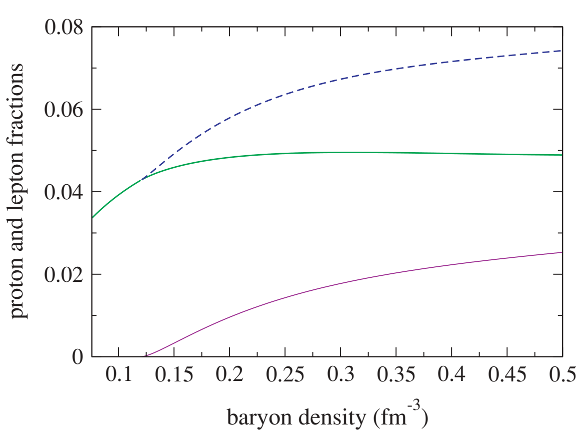

FIG. 1. (Color online) Equilibrium fractions of protons (dashed line), electrons (thick line), and muons (thin line) in neutron star liquid core as a function of the baryon density $n_b = n_p + n_n$ from the bottom edge of the crust $n_{\rm edge} \simeq 0.076$ fm^{-3} down to $3n_s$, where $n_s = 0.16$ fm^{-3} is the nuclear saturation density. The results were obtained with the Skyrme SLy4 energy density functional.

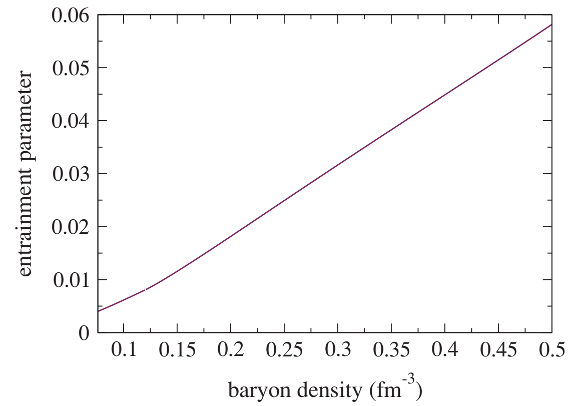

FIG. 2. (Color online) Dimensionless entrainment parameter ϵ for $npe\mu$ matter in β equilibrium (Skyrme SLy4 energy functional).

energies,

$$\begin{aligned}\mathcal{E}_0\{n_n, n_p\} = n_b m c^2 + \left(\frac{\hbar^2}{2m} + C_0^\tau n_b\right)\frac{3}{5}(3\pi^2)^{2/3}\left(n_n^{5/3} + n_p^{5/3}\right) \\ + C_0^n n_b^2 + C_1^n (n_n - n_p)^2 + C_1^\tau (n_n - n_p) \\ \times \frac{3}{5}(3\pi^2)^{2/3}\left(n_n^{5/3} - n_p^{5/3}\right),\end{aligned} \tag{80}$$

and $\mathcal{E}_{\rm ent}$ is the entrainment contribution expressible as

$$\mathcal{E}_{\rm ent} = -\tfrac{1}{2}\rho_n \varepsilon_n (\delta v)^2, \tag{81}$$

where $\boldsymbol{\delta v}$ is the velocity difference between neutrons and protons. The entrainment term is negligibly small compared to the static term $\mathcal{E}_{\rm ent} \ll \mathcal{E}_0$ even for the fastest pulsars and can therefore be neglected.

The muons are present in matter when the electron chemical potential μ_e exceeds the muon mass $m_\mu c^2 \simeq 105$ MeV. This occurs at a baryon density $n_b \simeq 0.12$ fm^{-3}. In equilibrium, the composition of the liquid core is therefore completely determined by the baryon density n_b (Fig. 1).

The dimensionless entrainment parameters as defined by (55) and (56), which have been widely used in neutron star simulations, are represented on Figs. 2 and 3, respectively.

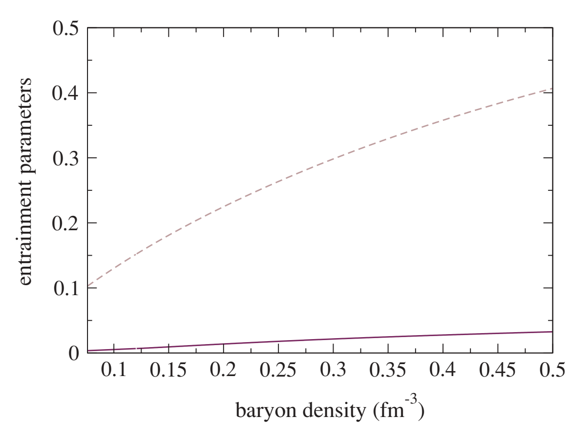

FIG. 3. (Color online) Dimensionless entrainment parameters ε_n (solid line) and ε_p (dashed line) for $npe\mu$ matter in β equilibrium (Skyrme SLy4 energy functional).

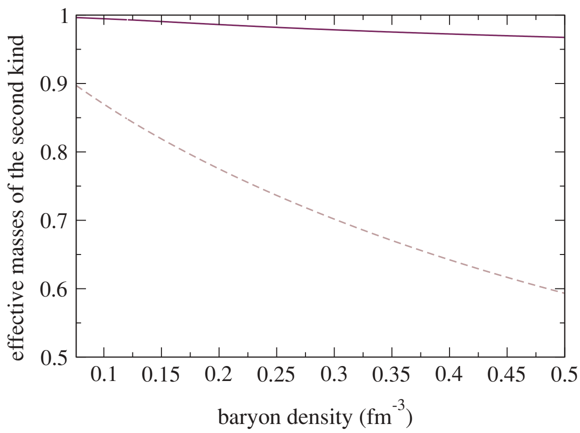

FIG. 4. (Color online) Effective masses $m^n_\star/m$ (solid line) and $m^p_\star/m$ (dashed line) for $npe\mu$ matter in β equilibrium (Skyrme SLy4 energy functional).

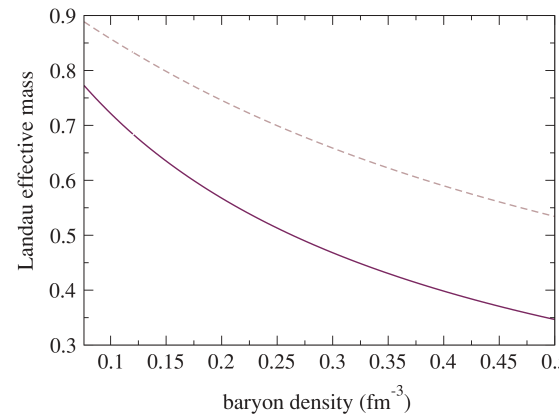

FIG. 6. (Color online) Landau effective masses $m^\oplus_n/m$ (solid line) and $m^\oplus_p/m$ (dashed line) for $npe\mu$ matter in β equilibrium (Skyrme SLy4 energy functional).

The $\star$ and $\sharp$ effective masses are shown on Figs. 4 and 5, respectively. Due to the increase of the proton fraction with the baryon density (see Fig. 1), the differences between the two kinds of effective masses $m^q_\star$ and $m^q_\sharp$, which are negligible at the crust-core boundary, become significant in deeper layers. We have also plotted the Landau effective masses for comparison in Fig. 6. As can be seen in those figures, the various definitions of "effective mass" do not coincide. This concept should therefore be carefully employed, and the definition that has been adopted should always be clearly specified.

Finally we show in Fig. 7 the dimensionless determinant of the entrainment matrix

$$\Upsilon = \frac{\rho_{nn}\rho_{pp} - \rho^2_{np}}{\rho_n\rho_p}, \tag{82}$$

which appears in the perturbed hydrodynamic equations and which is therefore important for the study of oscillation modes. In the present case, this quantity is simply given by

$$\Upsilon = 1 - \tilde{\alpha}_{np}\rho_b, \tag{83}$$

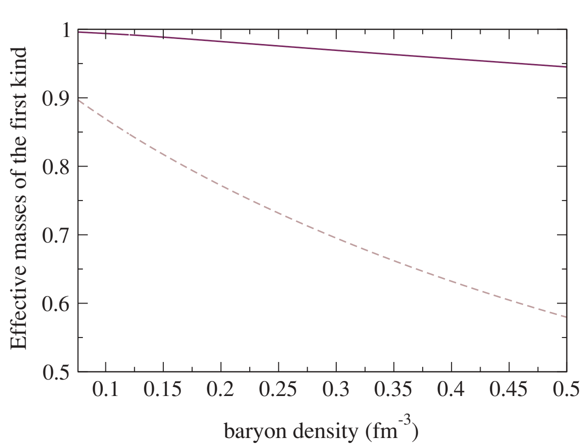

FIG. 5. (Color online) Effective masses $m^n_\sharp/m$ (solid line) and $m^p_\sharp/m$ (dashed line) for $npe\mu$ matter in β equilibrium (Skyrme SLy4 energy functional).

and is therefore quite remarkably independent of the nuclear asymmetry.

For comparison, we computed the coefficient $\tilde{\alpha}_{np}$ from which all the entrainment parameters can be obtained, for the 27 Skyrme forces recommended by Stone *et al.* [27] for neutron star studies. The coefficient $\tilde{\alpha}_{np}$ ranges from 0 for the parametrizations SkT4 and SkT5 [28] down to -10.4168 (in units m_p^{-1} fm^{-3}) for the parametrization SV [29]. Let us also mention that the Skyrme SLy forces [22–25], which have been widely employed in neutron star studies, yield coefficients around $\tilde{\alpha}_{np} \sim -1.5$ except for the SLy230a force, for which $\tilde{\alpha}_{np} = -0.007360$.

Notice that the three forces SkT4, SkT5, and SLy230a are the only parametrizations for which the isovector effective mass, relevant to the $T = 1$ isovector electric dipole sum rule, was set equal to the bare nucleon mass (see Table VII of Ref. [27]). However, the isovector electric dipole "giant resonance" in nuclei consists essentially of relative motion of protons against neutrons, and the sum rule constraint is therefore crucial for the entrainment effect in the infinite neutron-proton mixture. The case $\tilde{\alpha}_{np} = 0$ implies that the effective masses $m^q_\sharp$ and $m^q_\star$ are equal to the bare one, and

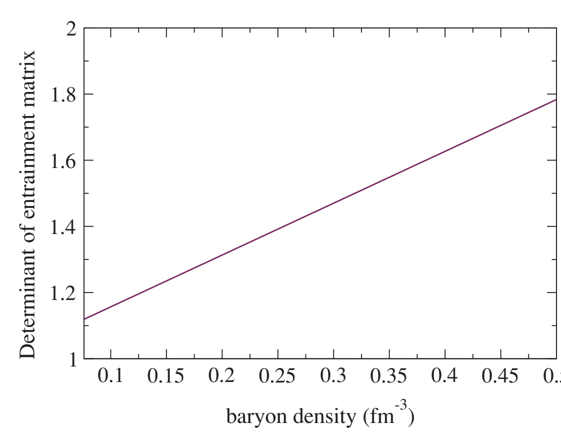

FIG. 7. (Color online) Dimensionless determinant Υ of the entrainment matrix (Skyrme SLy4 energy functional).

therefore there is no entrainment. Reciprocally large negative values of $\tilde{\alpha}_{np}$ are associated with strong entrainment effects.

In the present paper, we considered only spin-unpolarized nuclear matter. We therefore did not discuss spin and spin-isospin instabilities that plague many Skyrme forces at supranuclear densities [30,31]. However, notice that for the SLy4 used in our calculations, the ferromagnetic instability appears above the baryon density of 0.5 fm^{-3}, which is beyond the upper limit in our figures.

VII. CONCLUSION

Analytical expressions for the entrainment matrix and related effective masses of a neutron-proton superfluid mixture at zero temperature have been obtained within the nonrelativistic energy density functional theory. In contrast to recent investigations within relativistic mean field models [4], the entrainment parameters have been found to be expressible by very simple formulas which could be easily implemented in dynamical simulations of neutron star cores. We have also clarified the link between the various definitions of effective masses that have been introduced in the literature.

We have applied these formulas for Skyrme forces in order to evaluate the entrainment matrix in the standard model of the liquid core of neutron stars, composed of a mixture of neutrons, protons, electrons, and possibly muons in β equilibrium. In comparing the results with different Skyrme forces, we have found that the entrainment parameters are quite sensitive to the adopted parametrization. The observations of the entrainment effects in neutron stars could therefore provide new constraints on the construction of phenomenological nucleon-nucleon interactions and shed light on the properties of strongly asymmetric nuclear matter.

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APPENDIX: SKYRME ENERGY DENSITY FUNCTIONAL COEFFICIENTS

The energy functional deduced from the Skyrme effective interaction in the Hartree-Fock approximation has a form similar to Eq. (19). The coefficients in the energy functional (19) can thus be expressed in terms of the parameters of the Skyrme interaction as follows. As a result of the local phase invariance of the Skyrme forces [32], the coefficients C^j_T and C^τ_T are related by

$$C^j_T = -C^\tau_T. \tag{A1}$$

In terms of the parameters of the Skyrme interaction, the coefficients of the energy functional are given by [10]

$$C^n_0(n_b) = \tfrac{3}{8}t_0 + \tfrac{3}{48}t_3 n_b^\gamma, \tag{A2}$$

$$C^n_1(n_b) = -\tfrac{1}{4}t_0\left(\tfrac{1}{2}+x_0\right) - \tfrac{1}{24}t_3\left(\tfrac{1}{2}+x_3\right)n_b^\gamma, \tag{A3}$$

$$C^\tau_0 = \tfrac{3}{16}t_1 + \tfrac{1}{4}t_2\left(\tfrac{5}{4}+x_2\right), \tag{A4}$$

$$C^\tau_1 = -\tfrac{1}{8}t_1\left(\tfrac{1}{2}+x_1\right) + \tfrac{1}{8}t_2\left(\tfrac{1}{2}+x_2\right). \tag{A5}$$

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