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On the Lie subalgebra of Killing–Milne and Killing–Cartan vector fields in Newtonian spacetime

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The Galilean (and more generally Milne) invariance of Newtonian theory allows for Killing vector fields of a general kind, whereby the Lie derivative of a field is not required to vanish but only to be cancellable by some infinitesimal Galilean (respectively Milne) gauge transformation. In this paper, it is shown that both the Killing–Milne vector fields, which preserve the background Newtonian spacetime structure and the Killing–Cartan vector fields, which in addition preserve the gravitational field, form a Lie subalgebra.

Keywords: Killing vector; Newton–Cartan spacetime; Lie algebra.

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1. Introduction

Whereas the concept of Killing vector fields has been widely used in the general relativity theory to derive conservation laws from spacetime symmetries (see e.g. Ref. 1), its application to the Newtonian context is less well known. This mainly stems from the fact that Newtonian mechanics has been traditionally formulated using an "Aristotelian" decomposition of spacetime, as a direct product of a flat Euclidean three-dimensional space with a one-dimensional Euclidean timeline. This is only after Einstein proposed his theory of general relativity that a four-dimensional geometric formulation of Newtonian theory was developed by Cartan[2] (see e.g. Refs. 3 and 4 for a review). This formulation has been recently extended so as to include hydrodynamics[5–8] (allowing for fluid and superfluid mixtures), elasticity[9] and elasto-hydrodynamics.[10]

In Einstein's theory of general relativity, the occurrence of spacetime symmetries implies the vanishing of the Lie derivative of the Riemannian metric $g_{\mu\nu}$ (using Greek letters $\mu, \nu = 0, 1, 2, 3$ for spacetime indices) along one or several symmetry generators k_a^μ, $a = 0, 1, \ldots$ (see e.g. Ref. 1). Although the Lie algebras of the corresponding Killing vector fields in Newtonian spacetime have been already studied (see e.g. Ref. 11), the invariance of the physical laws of motion under Galilean (and more generally Milne) transformations allows for Killing vector fields of a more general kind, whereby the Lie derivative of a field is only required to be cancellable by some infinitesimal Galilean (respectively Milne) gauge transformation.[7] Two different kinds of such Killing vector fields were introduced in Ref. 7: the Killing–Milne vectors that preserve the Milne background structure of Newtonian spacetime and the Killing–Cartan vectors that in addition leave the gravitational field invariant.

In this paper, we shall demonstrate that the Killing–Milne and Killing–Cartan vector fields introduced in Ref. 7 form a Lie subalgebra after briefly reviewing the structure of the Newtonian spacetime and discussing the properties of these generalized Killing vector fields.

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2. Newtonian Spacetime Structure

Let us first briefly recapitulate the geometric structure of the Newtonian spacetime. We shall adopt the same notations as in Ref. 7. The existence of a universal time t leads to a foliation of the manifold into flat three-dimensional spaces. The pushforward of the three-dimensional Euclidean metric γ^{ij} (using Roman letters $i, j = 1, 2, 3$ for space indices) yields a symmetric contravariant tensor $\gamma^{\mu\nu}$ in the four-dimensional spacetime. This tensor itself is not metric since

$$\gamma^{\mu\nu} t_\nu = 0, \tag{1}$$

where $t_\nu = \partial t/\partial x^\nu \equiv \partial_\nu t$. The tensors $\gamma^{\mu\nu}$ and t_μ specifies the so-called Coriolis structure of Newtonian spacetime. A four-dimensional symmetric covariant tensor $\gamma_{\mu\nu}$ can be obtained by pulling back the Euclidean three-dimensional metric γ_{ij}. The degeneracy condition,

$$\gamma_{\mu\nu} e^\nu = 0, \tag{2}$$

implies the existence of a so-called "ether" frame flow vector e^μ, whose normalization can be chosen such that

$$e^\mu t_\mu = 1. \tag{3}$$

The vector e^μ characterizes a particular Aristotelian coordinate system $\{t, X^i\}$, in which $e^0 = 1$ and $e^i = 0$ corresponding to the usual kind of spacetime decomposition. The flatness of the three-dimensional hypersurfaces entails the existence of a natural connection, whose components $\Gamma^\rho_{\mu\nu} = 0$ vanish identically in the corresponding Aristotelian coordinate system (in other words, the covariant derivative ∇_μ is identifiable with the partial derivative ∂_μ). In an arbitrary coordinate system, some components of the connection may be nonzero. However, the associated covariant derivative should satisfy the following conditions:

$$\nabla_\rho \gamma^{\mu\nu} = 0, \quad \nabla_\rho \gamma_{\mu\nu} = 0, \quad \nabla_\mu t_\nu = 0, \quad \nabla_\mu e^\nu = 0. \tag{4}$$

As first shown by Cartan,[2] the gravitational vector field, defined by

$$g^\mu = -\gamma^{\mu\nu}\nabla_\nu \phi, \tag{5}$$

ϕ denoting the Newtonian gravitational potential, can be absorbed in a gravitationally modified connection

$$\omega^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - t_\mu g^\rho t_\nu. \tag{6}$$

The vector e^μ hence the tensor $\gamma_{\mu\nu}$ and the connection $\Gamma^\rho_{\mu\nu}$ are not uniquely defined. The *physical* structure of the Newtonian spacetime is preserved by Galilean transformations

$$e^\mu \to \breve{e}^\mu = e^\mu + b^\mu, \tag{7}$$

where b^μ is a spacelike boost velocity vector, whose spatial components b^i in an Aristotelian coordinate system are independent of the spatial coordinates X^i and of the time t. As first realized by Milne,[12] the Newtonian laws of motion in the presence of gravity are actually invariant under a more generic kind of transformations, whereby the spatial components b^i are allowed to depend on t, provided that the gravitational vector field be transformed as $g^i \to \breve{g}^i = g^i - a^i$ with $a^i = db^i/dt$. In an arbitrary coordinate system, the boost velocity vector will thus be required to satisfy[6]

$$t_\mu b^\mu = 0, \quad \gamma^{\nu\rho}\nabla_\rho b^\mu = 0. \tag{8}$$

Likewise, the relative acceleration vector field will be given by[6]

$$a^\mu = e^\nu \nabla_\nu b^\mu, \tag{9}$$

so that the gravitational vector field transforms as follows:

$$g^\mu \to \breve{g}^\mu = g^\mu - a^\mu. \tag{10}$$

The invariance of the Newtonian theory with respect to Milne transformations (10) is embedded in the invariance of the Newton–Cartan connection $\breve{\omega}^\rho_{\mu\nu} = \omega^\rho_{\mu\nu}$.

3. Killing–Milne and Killing–Cartan Vector Fields

The gauge invariance leads to symmetry generators k^μ_a of a new kind[7] such that the corresponding Lie derivative of any (gauge-dependent) field q is only required to vanish modulo some infinitesimal gauge transformation $\breve{d}_a q$:

$$\pounds_a q + \breve{d}_a q = 0, \tag{11}$$

where $\pounds_a \equiv \mathbf{k}_a \pounds$ denotes the Lie differentiation operator. Symmetry generators that preserve the background spacetime structure, namely the tensors t_μ, $\gamma^{\mu\nu}$ and

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e^μ (or equivalently t_μ, $\gamma^{\mu\nu}$ and the flat connection $\Gamma^\rho_{\mu\nu}$) are termed Killing–Milne vector fields.[7] Since only e^μ (or equivalently $\Gamma^\rho_{\mu\nu}$) are gauge-dependent, the Killing–Milne equations are

$$\pounds_a t_\mu = 0, \quad \pounds_a \gamma^{\mu\nu} = 0, \quad \pounds_a e^\mu + \check{d}_a e^\mu = 0. \tag{12}$$

These conditions lead to the following equations[7]:

$$t_\nu \nabla_\mu k_a^\nu = 0, \tag{13}$$

$$\gamma^{\rho(\mu}\nabla_\rho k_a^{\nu)} = 0, \tag{14}$$

$$e^\nu \nabla_\nu k_a^\mu = b_a^\mu, \tag{15}$$

where b_a^μ is the relevant boost velocity vector field and we have used brackets to indicate index symmetrization. A symmetry generator is termed Killing–Cartan vector field[7] if it also preserves the gravitational field g^μ

$$\pounds_a g^\mu + \check{d}_a g^\mu = 0, \tag{16}$$

or equivalently the (gauge-independent) Newton–Cartan connection $\omega^\rho_{\mu\nu}$

$$\pounds_a \omega^\rho_{\mu\nu} = 0. \tag{17}$$

Such a distinction between Killing–Milne and Killing–Cartan vector fields does not arise in the theory of general relativity since the invariance of the Riemannian metric automatically ensures the invariance of the gravitational field. The condition (16) or (17) yields[7]

$$D_\mu D_\nu k_a^\rho = -R_{\sigma\mu}{}^\rho{}_\nu k_a^\sigma, \tag{18}$$

or in terms of the gravitational potential,

$$e^\mu D_\mu \beta_a = -k_a^\mu D_\mu \phi, \tag{19}$$

where D_μ is the Newton–Cartan covariant derivative and $R_{\sigma\mu}{}^\rho{}_\nu$ the curvature tensor.[6] Let us remark that Eqs. (13) and (14) can be equivalently expressed as

$$t_\nu D_\mu k_a^\nu = 0, \tag{20}$$

$$\gamma^{\rho(\mu} D_\rho k_a^{\nu)} = 0. \tag{21}$$

This latter equation resembles Killing's equation in Riemannian spacetimes[1]:

$$D_{(\mu} k_{a\,\nu)} = 0, \tag{22}$$

where D_μ is the covariant derivative compatible with the metric.

It immediately follows from (22) that $k_{a\,\nu} u^\nu$ is conserved along the geodesic with tangent vector u^ν.[1] The proof is straightforward:

$$u^\mu D_\mu (k_{a\,\nu} u^\nu) = u^\mu u^\nu D_\mu k_{a\,\nu} + k_{a\,\nu} u^\mu D_\mu u^\nu = 0, \tag{23}$$

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the first term vanishes from (22) and the second from the geodesic equation. In Newtonian spacetime, we shall prove that the corresponding conserved quantity is

$$\mathcal{B}_a \equiv k_a^\mu \pi_\mu - \beta_a, \tag{24}$$

where

$$\pi_\mu \equiv v_\mu - \left(\frac{1}{2}v^2 + \phi\right) t_\mu, \quad v_\mu \equiv \gamma_{\mu\nu} u^\nu, \quad v^2 \equiv \gamma_{\mu\nu} u^\mu u^\nu = v_\mu u^\mu \tag{25}$$

and β_a is the boost potential defined by[6]

$$b_a^\mu = \gamma^{\mu\nu} \nabla_\nu \beta_a, \tag{26}$$

with b_a^μ given by Eq. (15). Let us write the derivative of $\mathcal{B}_a$ along a geodesic with tangent vector u^μ:

$$\begin{aligned} u^\mu D_\mu \mathcal{B}_a = {} & u^\mu v_\nu D_\mu k_a^\nu + k_a^\nu u^\mu D_\mu v_\nu - \frac{1}{2} t_\nu k_a^\nu u^\mu D_\mu v^2 - \frac{1}{2} v^2 t_\nu D_\mu k_a^\nu \\ & - t_\nu k_a^\nu u^\mu D_\mu \phi - \phi u^\mu t_\nu D_\mu k_a^\nu - u^\mu D_\mu \beta_a. \end{aligned} \tag{27}$$

Although $u^\mu D_\mu u^\nu$ vanishes from the geodesic equation, $u^\mu D_\mu v_\nu$ does not and is given by

$$u^\mu D_\mu v_\nu = u^\mu u^\rho D_\mu \gamma_{\nu\rho} = v_\mu g^\mu t_\nu + \gamma_{\mu\nu} g^\mu, \tag{28}$$

where we have used Eqs. (4) and (6) and the normalization $u^\mu t_\mu = 1$. Likewise, we find

$$u^\mu D_\mu v^2 = 2 g^\mu v_\mu. \tag{29}$$

Using Eqs. (28) and (29) as well as (20) in (27), we obtain

$$u^\mu D_\mu \mathcal{B}_a = u^\mu v_\nu D_\mu k_a^\nu + k_a^\nu g_\nu - t_\nu k_a^\nu u^\mu D_\mu \phi - u^\mu D_\mu \beta_a, \tag{30}$$

where $g_\nu \equiv \gamma_{\mu\nu} g^\mu$. Introducing the spacelike vector field $v^\mu \equiv u^\mu - e^\mu$ and using Eq. (15), Eq. (30) can be written as

$$u^\mu D_\mu \mathcal{B}_a = v^\mu v_\nu D_\mu k_a^\nu + k_a^\nu g_\nu - t_\nu k_a^\nu e^\mu D_\mu \phi - e^\mu D_\mu \beta_a. \tag{31}$$

The first term vanishes from the condition (21) that qualifies k_a^μ as a Killing–Milne vector field and the remaining terms cancel each other from Eq. (19) that qualifies k_a^μ as a Killing–Cartan vector field. We have thus proved that $\mathcal{B}_a$ is conserved along a geodesic:

$$u^\mu D_\mu \mathcal{B}_a = 0. \tag{32}$$

The maximum number of linearly independent Killing–Cartan vector fields can be determined along the same line of reasoning as in Riemannian spacetimes.[1] If k_a^μ and $K_\nu^\mu \equiv D_\nu k_a^\mu$ are known at some point $\mathcal{P}$, k_a^μ and K_ν^μ can be calculated at

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any other point $\mathcal{Q}$ by integrating the following system of equations along any curve connecting $\mathcal{P}$ and $\mathcal{Q}$

$$\xi^\mu D_\mu k_a^\nu = \xi^\mu K_\mu^\nu, \quad \xi^\mu D_\mu K_\nu^\rho = -\xi^\mu R_{\sigma\mu}{}^\rho{}_\nu k_a^\sigma, \tag{33}$$

where ξ^μ is the appropriate vector field and we have used Eq. (18). The number of linearly independent Killing–Cartan vector fields is therefore equal to the number of initial data, namely the components of k_a^μ and K_ν^μ at point $\mathcal{P}$. Equations (20) and (21), which can be expressed as $t_\nu K_\mu^\nu = 0$ and $\gamma^{\rho(\mu} K_\rho^{\nu)} = 0$, imply that K_μ^ν has only six independent components. With the four components of k_a^μ, we can thus conclude that the Newton–Cartan spacetime possesses at most ten linearly independent Killing–Cartan vector fields.

4. Lie Subalgebra of Killing Vector Fields

The identity (see e.g. Ref. 13)

$$[\boldsymbol{\xi}_a, \boldsymbol{\xi}_b]\pounds\ q = \boldsymbol{\xi}_a\pounds\ \{\boldsymbol{\xi}_b\pounds\ q\} - \boldsymbol{\xi}_b\pounds\ \{\boldsymbol{\xi}_a\pounds\ q\} \tag{34}$$

for any vector fields $\boldsymbol{\xi}_a$ and $\boldsymbol{\xi}_b$ automatically ensures that Killing vector fields of the usual kind (whereby the Lie derivative of a field is required to vanish) form a Lie subalgebra with the Lie bracket of two Killing vector fields $\mathbf{k}_a$ and $\mathbf{k}_b$ defined by their commutator

$$[k_a, k_b]^\mu = k_a^\nu \nabla_\nu k_b^\mu - k_b^\nu \nabla_\nu k_a^\mu = \pounds_a k_b^\mu, \tag{35}$$

where the last equality follows from the properties of the Lie derivatives (see e.g. Ref. 13). We shall now demonstrate that Killing vector fields of the generic kind form also a Lie subalgebra with the same definition of the Lie bracket.

Let us consider the successive action of an infinitesimal gauge transformation and the Lie differentiation of a gauge-dependent field q with respect to an arbitrary vector field $\boldsymbol{\xi}$. Under a change of gauge $e^\mu \to \breve{e}^\mu = e^\mu + b^\mu$, the ensuing fields $\breve{q}$ will be either a function of the boost velocity b^μ (for q fields like $\gamma_{\mu\nu}$) or of the corresponding acceleration $a^\mu = e^\nu \nabla_\nu b^\mu$ (for q fields like g^μ and $\Gamma^\rho_{\mu\nu}$). We shall treat these two cases separately. For q fields of the first kind, an infinitesimal gauge transformation is defined by

$$\breve{d}q = b^\mu \frac{\partial \breve{q}}{\partial b^\mu}, \tag{36}$$

where it is understood that the partial derivative is evaluated in the limit of vanishing boost velocity vector field $b^\mu \to 0$. By a suitable choice of coordinates adapted to the vector field $\boldsymbol{\xi}$, the Lie derivative reduces to a partial derivative to some coordinate x^1 (see e.g. Ref. 1): $\boldsymbol{\xi}\pounds = \partial/\partial x^1$. In this coordinate system, the Lie derivative of the field $\breve{d}q$ is thus simply given by

$$\boldsymbol{\xi}\pounds\ \breve{d}q = \frac{\partial b^\mu}{\partial x^1}\frac{\partial \breve{q}}{\partial b^\mu} + b^\mu \frac{\partial^2 \breve{q}}{\partial x^1 \partial b^\mu}. \tag{37}$$

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Likewise, the infinitesimal gauge transformation of the field $\boldsymbol{\xi}\pounds q$ is given by

$$\breve{d}\{\boldsymbol{\xi}\pounds\ q\} = b^\mu \frac{\partial^2 \breve{q}}{\partial x^1 \partial b^\mu}. \tag{38}$$

This shows that the Lie differentiation and the gauge transformation do not commute:

$$[\boldsymbol{\xi}\pounds\ , \breve{d}]q = \boldsymbol{\xi}\pounds\ \breve{d}q - \breve{d}\{\boldsymbol{\xi}\pounds\ q\} = (\boldsymbol{\xi}\pounds\ b^\mu)\frac{\partial \breve{q}}{\partial b^\mu}. \tag{39}$$

Although the field $\breve{q}$ depends on the boost velocity b^μ corresponding to the specific gauge transformation $e^\mu \to \breve{e}^\mu = e^\mu + b^\mu$, its *functional form* $\breve{q}\{b^\mu\}$ is actually gauge-independent. In other words, the fields $\breve{q}_a$ and $\breve{q}_b$ obtained from the same field q by the gauge transformations $e^\mu \mapsto \breve{e}^\mu = e^\mu + b^\mu_a$ and $e^\mu \mapsto \breve{e}^\mu = e^\mu + b^\mu_b$ respectively, are such that $\breve{q}_a\{b^\mu_a\} = \breve{q}\{b^\mu_a\}$ and $\breve{q}_b\{b^\mu_b\} = \breve{q}\{b^\mu_b\}$. Consequently, $\partial\breve{q}/\partial b^\mu$ is independent of b^μ. It can thus be seen from Eq. (36) that the commutator (39) represents an infinitesimal gauge transformation of the field q with a boost velocity vector field given by $\boldsymbol{\xi}\pounds b^\mu$.

The two successive gauge transformations $e^\mu \mapsto \breve{e}^\mu = e^\mu + b^\mu_a$ and $e^\mu \mapsto \breve{e}^\mu = e^\mu + b^\mu_b$ are obviously equivalent to the gauge transformation $e^\mu \mapsto \breve{e}^\mu = e^\mu + b^\mu_a + b^\mu_b$. As a consequence, the commutator of two infinitesimal gauge transformations vanishes:

$$[\breve{d}_a, \breve{d}_b]q = 0. \tag{40}$$

Combining Eqs. (39), (40) and (34) we finally find

$$[\boldsymbol{\xi}_a\pounds\ + \breve{d}_a, \boldsymbol{\xi}_b\pounds\ + \breve{d}_b] = \boldsymbol{\xi}_c\pounds\ + \breve{d}_c, \tag{41}$$

where we have introduced the vector field $\boldsymbol{\xi}_c \equiv [\boldsymbol{\xi}_a, \boldsymbol{\xi}_b]$ and the infinitesimal gauge transformation $\breve{d}_c$ is associated with the boost velocity vector field,

$$b^\mu_c \equiv \boldsymbol{\xi}_a\pounds\ b^\mu_b - \boldsymbol{\xi}_b\pounds\ b^\mu_a. \tag{42}$$

A similar analysis can be carried out for $\breve{q}$ fields that depend on a^μ rather than b^μ. In this case, an infinitesimal gauge transformation is defined by

$$\breve{d}q = a^\mu \frac{\partial \breve{q}}{\partial a^\mu}, \tag{43}$$

where $a^\mu = e^\nu \nabla_\nu b^\mu$ and the partial derivative is to be evaluated in the limit $a^\mu \to 0$. The functional form of $\breve{q}\{a^\mu\}$ is gauge independent therefore $\partial\breve{q}/\partial a^\mu$ is independent of a^μ. Proceeding as previously, Eq. (41) is found to still hold with the infinitesimal gauge transformation $\breve{d}_c$ associated with the boost acceleration vector field,

$$a^\mu_c \equiv \boldsymbol{\xi}_a\pounds\ a^\mu_b - \boldsymbol{\xi}_b\pounds\ a^\mu_a. \tag{44}$$

Let us now consider that $\boldsymbol{\xi}_a$ and $\boldsymbol{\xi}_b$ are Killing–Milne or Killing–Cartan vector fields. Using Eqs. (13), (14) and (15), we can show that $a^\mu_c = e^\nu \nabla_\nu b^\mu_c$. This means that the infinitesimal gauge transformation $\breve{d}_c$ acting on fields $\breve{q}\{b^\mu\}$ is the *same* as the infinitesimal gauge transformation $\breve{d}_c$ acting on fields $\breve{q}\{a^\mu\}$. The identity (41) thus implies that if $\mathbf{k}_a$ and $\mathbf{k}_b$ are Killing vector fields, their commutator

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$\mathbf{k}_c = [\mathbf{k}_a, \mathbf{k}_b]$ is also a Killing vector field. Using Eqs. (4) and (42), it can be checked that the boost velocity vector field b_c^μ associated with $\mathbf{k}_c$ is given by

$$b_c^\mu = e^\nu \nabla_\nu [\mathbf{k}_a, \mathbf{k}_b]^\mu = e^\nu \nabla_\nu k_c^\mu, \tag{45}$$

in accordance with Eq. (15).

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